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Perspectives in Measurement, Modeling and Interpretation

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Wimmer Gejza is a senior researcher at the Mathematical Institute of the Slovak Academy of Sciences in Bratislava. He belongs to the most prominent statisticians and metrologists in Slovakia. He has made a substantial contribution in development of statistical models and algorithms for calibration of measurement devices, statistical methods for inter-laboratory comparisons evaluation and expression of the measurement uncertainties. His research focuses on development of a comprehensive theory of measurement devices calibration, establishment of the relevant uncertainties in the comparative calibration of sensors and other measurement instruments as well as evaluation of the measurement results.Professor Wimmer was successful in combining the standard statistical modeling approach, based on a proper modeling of the measured data, and the approach based on the metrology principles. He is an author or a co-author of seven monographs, seven book chapters, and of more than 110 scientific papers, cited more than 900 times. He was principal investigator or investigator of 16 research projects and participated in editorial boards of 5 international scientific journals.

Witkovský Viktor is a senior scientist at the Institute of Measurement Science of the Slovak Academy of Sciences in Bratislava (IMS SAS), a mathematical statistician with scientific research experience focused on statistical methods for linear models with fixed and random effects, small sample theory development, research methods and algorithms for calculation of the exact and approximate probability distributions, theory development in measurement science and metrology as well as applied statistical research in the biomedical field. He served as investigator or coordinator of 15 research projects (EC FP6, APVV, VEGA). He is the author or co-author of 50 scientific articles ,of three domestic monographs and more than 60 conference papers, presentations and other publications with more than 450 citations, member of editorial boards of scientific journals (Measurement Science Review, Journal of Breath Research, Acta et Commentationes Universitatis Tartuensis de Mathematica Colloquium Biometricum), and member in international organizations (International Association of Breath Research). He is active in the creation of international standards ISO, as well as regular assessment of research projects, scientific articles, and the review of report magazines (e.g. Zentralblatt MATH, Mathematical Reviews)

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Žáková Myšková Kateřina is an Assistant Professor at the Department of Statistics and Operation Analysis of the Mendel University in Brno. She dealt with the multidimensional calibration model in her dissertation. She tested the use of calibration models on economic data. Currently, her research area is the economic and ecological analysis of environmental measures. She is involved in research of optimization of motorways permeability for wildlife, efficient and environmental use of biochar in constructions, use of Vegetation Greenery Systems for urban heating mitigation and External Thermo Insulation Composite Systems optimization. She published several important (much-cited) publications in impacted journals in this area. She was also a co-investigator of several ecological projects (Eureka, TACR).

Foreword

The present publication is the next one of a series of books inspired by presentations at the International Workshop on Knowledge Management (IWKM). The volumes do not appear in regular intervals but they mark the IWKM's history quite well.

The workshop came in life in 2006 as a natural reaction to the introduction and development of the Knowledge Management study program at Vysoká škola manažmentu (School of Management) in Trenčín, Slovakia. The workshop followed two main aims:

(a) To present the concepts of knowledge management to the wider Slovak audience;
(b) The invite researchers to this (then rather young) field of science and to give them room for their publications.

The workshops ran annually. In addition to their regular proceedings, selected publications were collected and edited into monographs like this one. In the first two years, the workshop was a national event. That's why the first volume is in Slovak [1]. As the idea of organizing such an event proved itself a success, the workshop organizers internationalized it and switched its lingua franca from Slovak to English. Between 2008 and 2010, the workshop was incorporated into a two-week long Summer School on Knowledge Management sponsored by the Erasmus program. The students and lecturers from the Czech Republic, Finland, Lithuania and Slovakia participated and collaborated here. Their outcomes are monographs [2, 3, 4].

This edited book is a free continuation of the series and represents a new level of our understanding of knowledge management concepts. Initially, knowledge was understood as a *tool* enhancing efficient and effective production. In the contemporary elucidation, it is an "independent", autonomous *value* which can – and must – be measured, evaluated and interpreted. Such an approach demonstrates a mental shift from a simple utilitarian approach to its depiction as a mental, intangible asset.

Due to this transformation on the principles, one can ask: "*What everything can be measured and how? And what can we learn from ability (or inability) to do it?*" Questions posed in this way open new horizons in knowledge management and start another era in its research. I hope that readers will enjoy this rich variety. The quantum of research directions proposes promising perspectives to everyone wishing to learn more about measuring the unknown and about excerpting knowledge from these measurements.

I therefore congratulate the editor and authors to their achievement. I am certain that every paper will find its readers who will get inspired by them and will extend this fresh IWKM heritage.

I wish you happy readings.

Jozef Hvorecký, IWKM co-founder
Bratislava, November 2020

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Introduction

The prominence of measurement for research rigor, applied science, commerce and other social utilization is indisputable. However, despite its conventional standing, frequent employment across disciplines, times and locations, remarkable advances in metrology and related regulation (standardization), various conceptual and practical challenges related to measurement persist (inclusive of its exact meaning).

The main objective of this edited volume is to consider issues and present ideas that are specific to the context of measurement from the perspectives of various disciplines and methodological approaches.

The target readership includes scholars and professionals interested in the problem of measurement and searching for related themes and ideas from a range of angles (the volume contains 21 authors from 14 institutions). Hopefully, this mutual sharing of ideas and experiences will also generate opportunities for further innovative inspirations and fruitful collaborations among involved researchers, professionals and establishments.

This publishing initiative follows up on some of the contributions and informal talks held at the 13th International Workshop on Knowledge Management (IWKM) organized by Vysoká škola manažmentu/City University of Seattle in Bratislava (October 18-19, 2018). The two of IWKM seminars discussed topics related to "Data Mining Tools and Measurement Data Analysis" and to "Measuring Quality of Universities' Functions".

The most frequent keywords of the seminar discourse had been measurement, modeling and interpretation and these seemed to be a natural thematic choice for this publishing enterprise.

The position of measurement as a source of data and knowledge aspires increasing number of scholars across disciplines to utilise this mode of enquiry in their research problems and solutions. This is done with varying levels of achievement and quality. It only seemed fitting to set up a forum for sharing original perspectives and experiences from quantification and related processes (e.g. interpretation of quantities).

The main point has been to create a room for mutual inspiration and learning from achievements of experts in diverse fields of research and practice. This initiative has tried to emphasise pragmatic suggestions and bottom lines. We believe that the fact that measurement is theory loaded and highly contextual should not make transfer of competences unmanageable, thus scholars from even very differing areas of exploration can benefit from sharing their original perspectives.

Problems and motivations in quantification can be very diverse and so is the character of this book. The scope of the disciplines the chapters make arguments from includes metrology, statistics, algorithmic graph theory, economics, business administration and knowledge management.

It is hard to see how one would solemnly opine on measurement without addressing very true authority on weights and measures. Hence this volume opens with the chapter from the most prominent explanation of measurement – metrology.

Gejza Wimmer, Viktor Witkovský and Kateřina Myšková Žáková in an important way contribute to the evaluation of measurement calibration. They put forward a new calibration model with linear calibration function, based on comparison of two measuring devices – each measuring two-dimensional measurements with normally distributed errors. The chapter gives a detailed account of the measuring process and statistical properties of the measurements from the calibrated device.

In chapter two, the researchers in the subgraph mining and the graphlet-based network analysis - Martin Nehéz, Miroslav Tatarko and Alexander Maťašovský present the concept of frequent subgraph mining (FSM) and explain its algorithmic principles. FSM algorithms are applicable in increasing number of research areas, their text focuses on the subgraph mining problem in chemical datasets and the graphlet-based comparisons of real-world networks. Nehéz et al. distinctly demonstrate FSM application in their current research problems: data mining in large chemical datasets using the ConQuest software and the graphlet-based analysis of protein-protein interaction networks.

In chapter three, Milan Terek considers prospects of information channel effectiveness evaluation using the statistical survey data. He analyses association between the categorical variables using chi - squared tests and residual analysis and measures association strength by odds ratio.

The proposed assessment model evaluates information channel effectiveness with various types of users and the chapter illustrates its application in two different contexts – academic and corporate.

As the author further suggests, the presented three step evaluation model can be used outside of information channels, for example serve as a tool for determining the association between hospitality satisfaction and various types of hotel quests.

In chapter four, Luboš Marek and Petr Berka apply descriptive and analytical ways to statistically analyze and model wage and income distributions in the Czech Republic.

In the descriptive part the authors evaluate the wage developments and trends for a period of 23 years. Their detailed empirical study includes precise gender and age considerations and yields interesting results. In the analytical passage Marek and Berka demonstrate differing probabilistic distributions employed in wage distribution models and tell how these models can be used to make wage distribution predictions.

In chapter five, Monika Šestáková argues that current business models of digital companies make many traditional concepts and definitions in taxation policies obsolete and measurement of international taxation constituents often impossible or not appropriate. The author calls for redefining some basic concepts in international taxation as digitalized businesses can control their activities in foreign countries without their physical presence. The crucial concepts and issues that should be redefined or considered include the permanent establishment; the key role of data in the strategy of digital companies (data collected from foreign sources and not paid for); the importance and high flexibility of intellectual capital; forms of outsourcing business functions and increasing role of e-commerce. The most important part of the text focuses on how the value created at the territory of a particular country can be defined and subsequently measured.

In chapter six, Sonia Ferenčíková, Jana Hrdličková, Janka Pasztorová and Daniel Krajcik discuss effects of knowledge transfer and the tools which can be used to quantify such effects. They specifically look at the corporate reverse knowledge transfer (transfer from subsidiary to parent company) in the cases of two sales & marketing companies in Slovakia. As they further claim, sharing of knowledge delivers multiple benefits and is an efficient tool for boosting competitiveness and global corporate performance. In respect to specifying and tracking knowledge transfer contributions, the authors suggest that both quantitative and qualitative impacts should be considered. While Ferenčíková et al. acknowledge that determining the effect of acquired knowledge implementation is quite demanding issue, the scoring systems covering all possible effects on corporate performance would be able to illustrate the full picture of knowledge effects on the company.

In chapter seven, Jozef Šimúth and Andrea Zacharová focus on possibilities of measuring the tacit and explicit knowledge. Important is their account of beehive approach to knowledge measurement in which they focus on the transformation of individual knowledge into organizational knowledge. This transformation makes knowledge visible in the organizational performance. As one of the reviewers puts it, measuring knowledge is a difficult task not only because it is invisible but also because its amount invisibly changes, so their efforts might go in vain if they would not act with a considerable patience. Their study of many sources and demonstration of a variety of approaches assuredly represents a serious contribution to the topic.

In chapter eight, Christian Schlicht contributes to solving theoretical and practical problems of measuring added value in the field of facility management. He develops a measuring model which is illustrated in the conditions of the leading German shopping center business and shows how the added value measurement can be transferred into operations of facility management.

In chapter nine, Mário Hegedüs writes about current trends in measuring innovation activities, shows shortcomings of the main methods and characterizes the current discourse on the issue.

He provides detailed analysis of the way how the basic and widely used methodology (Oslo Manual) responds to the new tendencies in the theory and practice of innovation activity. Although the chapter is concentrated on macroeconomic aspects of innovation measurement, it can be inspiring also for business organizations.

In chapter ten, Mária Tajtáková and Mária Olejárová analyse practices in knowledge transfer with a special attention to the role of knowledge brokers. They deal with the modeling of knowledge brokering which is a tool for facilitating the exchange of knowledge between its producers and users. Based on their qualitative research they explored culture-based urban regeneration processes applying Nonaka and Takeuchi's SECI model as well as the Knowledge Broker Intervention Model by Shaxson and Gwyn et al. The research covers revitalization projects at six organizations located in three cities in the Slovak Republic.

All in all, the IWKM seminars and following collaborations on this edited volume had provided platform for networking, sharing of lessons learned and discussion of best practices.

The formal contributions as well as informal debates showed general understanding of important role quantitative representations play in deeper comprehension of studied phenomena, no matter what research area or type of inquiry. The same applies to the task of measurement in making and testing predictions. On the other hand, it may be the case, that in popular discourse, the term ″measurement″ often gets clichéd or misapplied.

Hopefully the following chapters offer an interesting reading and contribute to the catalogue of perspectives in the area of quantification and interpretation of quantitative data.

I would like to extend my sincere gratitude and appreciation for all of the diligent work and dedication provided by the authors, reviewers and associates involved in the project.

Thank you!

Branislav Bernadič, Editor

I

Two-dimensional linear comparative calibration

Gejza Wimmer

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1 Introduction

We consider a *comparative calibration problem*, respecting the metrological point of view, which is different from the traditional approach based on comparing a new sensor (measuring device) with an accurate calibration standard, as frequently considered in analytical chemistry (see e.g. [3]) or in other scientific, technical, and/or economic fields (see e.g. [1]). According to Wikipedia, *calibration in measurement technology and metrology is the comparison of measurement values delivered by a device under test with those of a calibra-tion standard of known accuracy. Such a standard could be another measurement device of known accuracy, a device generating the quantity to be measured such as a voltage, sound tone, or a physical artefact, such as a metre ruler*. According to [9], calibration is defined as *an operation that, under specified conditions, in a first step, establishes a relation between the quantity values with measurement uncertainties provided by measurement standards and corresponding indications with associated measurement uncertainties (of the calibrated instrument or secondary standard) and, in a second step, uses this information to establish a relation for obtaining a measurement result from an indication.*

This chapter deals with two-dimensional comparative calibration, i.e. with a situation when two measuring devices, say $\mathcal{X}$ and $\mathcal{Y}$ (here we shall assume that $\mathcal{Y}$ is the calibration standard), which enter to the calibration process, and both of them indicate two-dimensional measurement results. Measurements realized by each measuring device $\mathcal{X}$ and $\mathcal{Y}$ are subject to errors. Of course, if we measure a two-dimensional quantity, we obtain only indication (evidence value, estimate) of that two-dimensional quantity.

In our approach, the calibration function expresses the relation between couples of the ideal true values of the quantities (objects) measured simultaneously by both measuring devices. In general, proper expression of the *calibration function* in a specific calibration task is an important and critical issue which requires understanding of the calibration process and is typically based on deep discussions of the involved metrologists, physicists, chemists, constructors of measuring devices, etc. Here, we shall assume that the calibration function is a linear function. But due to the errors in both couples of measurements (realized by measuring device $\mathcal{X}$ and $\mathcal{Y}$, respectively), the considered calibration model is nonlinear in the model parameters. As it will be presented below, the calibration model can be presented as an errors-in-variables model and after proper linearization as a linear model with linear restrictions on the parameters.

In Section 2.1 is established the two-dimensional comparative calibration model. It is supposed that the two-dimensional measurements (realized by the measuring devices $\mathcal{X}$ and $\mathcal{Y}$, respectively) are normally distributed. Moreover, we shall assume that the two-dimensional measurements are mutually independent with possibly unknown parameters of their covariance matrices. As already said, the calibration function is a linear function but the considered calibration model is nonlinear in its parameters. Thus, the nonlinear calibration model is linearized and then the *Best Linear Unbiased Estimators* (BLUEs) of the model parameters are derived, together with their covariance matrices. In Section 2.2 is introduced the replicated two-dimensional calibration model together with the BLUEs of the model parameters. Moreover, in the replicated model it is possible to estimate the unknown covariance parameters (covariance matrices of measurements realized by $\mathcal{X}$ and $\mathcal{Y}$) by using the *Minimum Norm Quadratic Unbiased Estimator* (MINQUE) procedure. In Section 2.3 are given the respective MINQUE estimators. In Section 2.4 is proposed an iterative procedure for estimating parameters of the (nonlinear) calibration model. In Section 2.5 we derived the (approximate) confidence regions for the whole vector of unknown model parameters (Section 2.5.1), for the individual parameters of the calibration function (Section 2.5.2), for special important linear functions of the model parameters (Section 2.5.3), and finally, the confidence region for the true unknown values of the measured couple by the less precise measuring device expressed in units of the standard. Section 2.6 derives the measuring process with the calibrated (less precise) measuring device. Appendices 1 to 4 present a brief summary of selected mathematical-statistical results with proper references, useful for better understanding of the procedures mentioned this text.

2 Two-dimensional comparative calibration model

2.1 Linear two-dimensional comparative calibration model

Let us consider the comparative calibration experiment with two measuring devices, say $\mathcal{Y}$ and $\mathcal{X}$, respectively. Here we shall consider two-dimensional calibration, i.e. each measuring device is indicating two-dimensional vector of measured values. Calibration (process) establishes a relation between the two-dimensional quantity values provided by the measuring device $\mathcal{Y}$ and corresponding two-dimensional values indicated on the measuring device $\mathcal{X}$. Here, we shall suppose that $\mathcal{Y}$ is the calibration standard and $\mathcal{X}$ is the measuring device to be calibrated.

Of course, if we measure a particular object, represented by a two-dimensional vector of quantities, using the measuring device $\mathcal{Y}$, we obtain only the indication (evidence value, estimate), say $\boldsymbol{y} = (y_1, y_2)'$, of the couple of true values $\boldsymbol{\nu} = (\nu_1, \nu_2)'$, expressed in units of the measuring device $\mathcal{Y}$. In the same way, if we measure the same object by the measuring device $\mathcal{X}$, we obtain only the indication (evidence value, estimate) $\boldsymbol{x} = (x_1, x_2)'$ of the true values $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ (expressed in the units of the measuring device $\mathcal{X}$). Frequently, it is desirable that the indications are expressed in the same units. In this paper, the theoretical calibration function, $\boldsymbol{\nu} = \boldsymbol{f}(\boldsymbol{\mu})$, is understood as a function which expresses the relation between the ideal (true, errorless) values of the same object measured by the measuring device $\mathcal{X}$ and $\mathcal{Y}$, respectively. More precisely we assume, that the calibration function is linear, i.e.

$$\boldsymbol{\nu} = \boldsymbol{f}(\boldsymbol{\mu}) = \boldsymbol{a} + \boldsymbol{B}\boldsymbol{\mu}. \tag{1}$$

So, let us consider that by the two measuring devices $\mathcal{X}$ and $\mathcal{Y}$ we measure a couple of values (two values, e.g. two coordinates) of n objects. Measurements with the measuring device $\mathcal{X}$ are two-dimensional random vectors $\boldsymbol{X}_i$, $i = 1, 2, \ldots, n$, and we assume that

$$\boldsymbol{X}_i = \begin{pmatrix} X_{1,i} \\ X_{2,i} \end{pmatrix} \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_X), \quad i = 1, 2, \ldots, n.$$

In particular, we assume that $\boldsymbol{X}_i$ has a two-dimensional normal distribution with mean $\boldsymbol{\mu}_i = \begin{pmatrix} \mu_{1,i} \\ \mu_{2,i} \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma}_X = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$.

Similarly, measurements with the measuring device $\mathcal{Y}$ are two-dimensional random vectors $\boldsymbol{Y}_i$, $i = 1, 2, \ldots, n$, and we assume that

$$\boldsymbol{Y}_i = \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \end{pmatrix} \sim N(\boldsymbol{\nu}_i, \boldsymbol{\Sigma}_Y), \quad i = 1, 2, \ldots, n,$$

with the mean $\boldsymbol{\nu}_i = \begin{pmatrix} \nu_{1,i} \\ \nu_{2,i} \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma}_Y = \begin{pmatrix} \sigma_3^2 & 0 \\ 0 & \sigma_4^2 \end{pmatrix}$. Further, we assume that measurements $\boldsymbol{X}_i$ and $\boldsymbol{Y}_j$ are independent for all i, j. It can be expressed as

$$\operatorname{cov}(\boldsymbol{X}_i, \boldsymbol{Y}_j) = \boldsymbol{0} \ \text{ for all } i, j = 1, 2, \ldots, n, \quad \operatorname{cov}(\boldsymbol{X}_i, \boldsymbol{X}_j) = \boldsymbol{0}, \quad \operatorname{cov}(\boldsymbol{Y}_i, \boldsymbol{Y}_j) = \boldsymbol{0} \ \text{ for } \ i \neq j.$$

The measurements can be arranged as matrices

$$\boldsymbol{X} = \begin{pmatrix} X_{1,1} & X_{1,2} & \ldots & X_{1,n} \\ X_{2,1} & X_{2,2} & \ldots & X_{2,n} \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \ldots & Y_{1,n} \\ Y_{2,1} & Y_{2,2} & \ldots & Y_{2,n} \end{pmatrix}$$

and in the same way the means

$$\boldsymbol{M} = \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \ldots & \mu_{1,n} \\ \mu_{2,1} & \mu_{2,2} & \ldots & \mu_{2,n} \end{pmatrix}, \qquad \boldsymbol{N} = \begin{pmatrix} \nu_{1,1} & \nu_{1,2} & \ldots & \nu_{1,n} \\ \nu_{2,1} & \nu_{2,2} & \ldots & \nu_{2,n} \end{pmatrix}.$$

We shall use the operator vec (vectorization of a matrix). This operator means a linear transformation which converts the matrix into a column vector (arranges the columns of the matrix in a natural order).

Now, the measurement model can be written as

$$\begin{pmatrix} \operatorname{vec} \boldsymbol{X} \\ \operatorname{vec} \boldsymbol{Y} \end{pmatrix} \sim \left[\begin{pmatrix} \operatorname{vec} \boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix}, \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \boldsymbol{0}_{2n} \\ \boldsymbol{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix} \right] \tag{2}$$

with (nonlinear) constraints on the parameters

$$\boldsymbol{\nu}_i = \boldsymbol{a} + \boldsymbol{B}\boldsymbol{\mu}_i \quad \text{ for } i = 1, 2, \ldots, n, \tag{3}$$

where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$. The notation (2) means that the mean of the random vector $\begin{pmatrix} \operatorname{vec} \boldsymbol{X} \\ \operatorname{vec} \boldsymbol{Y} \end{pmatrix}$ is $\begin{pmatrix} \operatorname{vec} \boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix}$ and its covariance matrix is $\begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \boldsymbol{0}_{2n} \\ \boldsymbol{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix}$, the symbol $\otimes$ means the Kronecker product. The constraints on the parameters can be written as

$$\boldsymbol{N} = \boldsymbol{1}_n' \otimes \boldsymbol{a} + \boldsymbol{B}\boldsymbol{M},$$

or

$$\operatorname{vec} \boldsymbol{N} = \mathbf{1}_n \otimes \boldsymbol{a} + (\boldsymbol{I}_m \otimes \boldsymbol{B}) \operatorname{vec} \boldsymbol{M} \tag{4}$$

where $\mathbf{1}_n$ is the $n-$dimensional vector $(1, 1, \ldots, 1)'$. The measurement model (2)-(4) is a nonlinear model in its parameters, because the constraints (4) depend on the product of parameters $\boldsymbol{B}$ and $\boldsymbol{M}$.

Next, we shall linearize the constraints (4) at proper values $\boldsymbol{B}_0$ and $\boldsymbol{M}_0$, by using the Taylor expansion with neglecting the term $(\boldsymbol{I}_n \otimes \delta\boldsymbol{B}) \operatorname{vec} \delta\boldsymbol{M}$ and the terms with higher order. We obtain a (linear) approximation of the constraints (4)

$$\operatorname{vec} \boldsymbol{N} \approx \mathbf{1}_n \otimes \boldsymbol{a} + (\boldsymbol{I}_n \otimes \boldsymbol{B}) \operatorname{vec} M_0 + (\boldsymbol{I}_n \otimes \delta\boldsymbol{B}) \operatorname{vec} \boldsymbol{M}_0 + (\boldsymbol{I}_n \otimes \boldsymbol{B}_0) \operatorname{vec} \delta\boldsymbol{M}, \tag{5}$$

where $\delta\boldsymbol{B} = \boldsymbol{B} - \boldsymbol{B}_0$, $\delta\boldsymbol{M} = \boldsymbol{M} - \boldsymbol{M}_0$. Based on Appendix 1, we get

$$(\boldsymbol{I}_n \otimes \delta\boldsymbol{B}) \operatorname{vec} \boldsymbol{M}_0 = \operatorname{vec}(\delta\boldsymbol{B}\ \boldsymbol{M}_0\ \boldsymbol{I}_n) = \operatorname{vec}(I_2\ \delta\boldsymbol{B}\ \boldsymbol{M}_0) = (\boldsymbol{M}_0' \otimes I_2) \operatorname{vec} \delta\boldsymbol{B}$$

and hence, we can rewrite the condition (5) to the following form

$$(\boldsymbol{I}_n \otimes \boldsymbol{B}_0) \operatorname{vec} \boldsymbol{M}_0 + (\boldsymbol{I}_n \otimes \boldsymbol{B}_0 \vdots - \boldsymbol{I}_{2n}) \begin{pmatrix} \operatorname{vec} \delta\boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix} + (\mathbf{1}_n \otimes \boldsymbol{I}_2 \vdots \boldsymbol{M}_0' \otimes \boldsymbol{I}_2) \begin{pmatrix} \boldsymbol{a} \\ \operatorname{vec} \delta\boldsymbol{B} \end{pmatrix} = \mathbf{0}. \tag{6}$$

Finally we obtain the linearized model

$$\begin{pmatrix} \operatorname{vec} \boldsymbol{X} - \operatorname{vec} \boldsymbol{M}_0 \\ \operatorname{vec} \boldsymbol{Y} \end{pmatrix} \sim \left[\begin{pmatrix} \operatorname{vec} \delta\boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix}, \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \mathbf{0}_{2n} \\ \mathbf{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix} \right] \tag{7}$$

with (linear) constraints (6) on $\delta\boldsymbol{M}$ and $\delta\boldsymbol{B}$, and the covariance matrix $\begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \mathbf{0}_{2n} \\ \mathbf{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix}$.

We shall call the model (7) with constraints (6) as the *non-replicated two-dimensional comparative calibration model*. In general, this model is also known as a linear model of measurements with type-II constraints (see e.g. [2], [5]). According to the Appendix 1, the following holds true:

Theorem 1. In the non replicated two-dimensional comparative calibration model (7)-(6) the BLUE (best linear unbiased estimators) of the parameters $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{a}$, and $\delta\boldsymbol{B}$ are given by

$$\operatorname{vec} \widehat{\boldsymbol{M}} = \operatorname{vec} \boldsymbol{X} + (\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1})(\operatorname{vec} \boldsymbol{Y} - \operatorname{vec} \boldsymbol{B}_0 \boldsymbol{X}), \tag{8}$$

$$\operatorname{vec} \widehat{\boldsymbol{N}} = \operatorname{vec} \boldsymbol{Y} - (\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1})(\operatorname{vec} \boldsymbol{Y} - \operatorname{vec} \boldsymbol{B}_0 \boldsymbol{X}), \tag{9}$$

$$\begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec} \widehat{\delta\boldsymbol{B}} \end{pmatrix} = \left(\begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_n' \\ \boldsymbol{M}_0 \end{pmatrix} \otimes \boldsymbol{I}_2 \right) (\operatorname{vec} \boldsymbol{Y} - \operatorname{vec} \boldsymbol{B}_0 \boldsymbol{X}). \tag{10}$$

The covariance matrix of these estimators are

$$\operatorname{cov} \begin{pmatrix} \operatorname{vec} \widehat{\boldsymbol{M}} \\ \operatorname{vec} \widehat{\boldsymbol{N}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \mathbf{0}_{2n} \\ \mathbf{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix} -$$

$$- \begin{pmatrix} \mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1} \boldsymbol{B}_0 \boldsymbol{\Sigma}_X & -\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1} \boldsymbol{\Sigma}_Y \\ -\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1} \boldsymbol{B}_0 \boldsymbol{\Sigma}_X & \mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1} \boldsymbol{\Sigma}_Y \end{pmatrix}, \tag{11}$$

$$\operatorname{cov} \begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec} \widehat{\delta\boldsymbol{B}} \end{pmatrix} = \begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \otimes \boldsymbol{C}, \tag{12}$$

$$\operatorname{cov} \left(\begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec} \widehat{\delta\boldsymbol{B}} \end{pmatrix}, \begin{pmatrix} \operatorname{vec} \widehat{\boldsymbol{M}} \\ \operatorname{vec} \widehat{\boldsymbol{N}} \end{pmatrix} \right) = \left(- \begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_n' \\ \boldsymbol{M}_0 \end{pmatrix} \otimes \boldsymbol{B}_0 \boldsymbol{\Sigma}_X, \right.$$

$$\begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_n' \\ \boldsymbol{M}_0 \end{pmatrix} \otimes \boldsymbol{\Sigma}_Y \Bigg), \tag{13}$$

where $\boldsymbol{C} = \boldsymbol{B}_0 \boldsymbol{\Sigma}_X \boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y$ and $\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} = \boldsymbol{I}_n - (\mathbf{1}_n, \boldsymbol{M}_0') \begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_n' \\ \boldsymbol{M}_0 \end{pmatrix}$. Detailed proof can be found in [7].

2.2 Replicated two-dimensional comparative calibration model

If we replicate the (whole) measurement experiment described in Section 2.1 independently m times, the replicated two-dimensional comparative calibration model can be written as

$$\begin{pmatrix} \operatorname{vec} \boldsymbol{X}^1 - \operatorname{vec} \boldsymbol{M}_0 \\ \operatorname{vec} \boldsymbol{Y}^1 \\ \vdots \\ \operatorname{vec} \boldsymbol{X}^m - \operatorname{vec} \boldsymbol{M}_0 \\ \operatorname{vec} \boldsymbol{Y}^m \end{pmatrix} \sim \left[(\mathbf{1}_m \otimes \boldsymbol{I}_{4n}) \begin{pmatrix} \operatorname{vec} \delta \boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix}, \boldsymbol{I}_m \otimes \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \mathbf{0}_{2n} \\ \mathbf{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix} \right] \tag{14}$$

where

$$\boldsymbol{X}^j = \begin{pmatrix} X_{1,1}^j & X_{1,2}^j & \dots & X_{1,n}^j \\ X_{2,1}^j & X_{2,2}^j & \dots & X_{2,n}^j \end{pmatrix}, \qquad \boldsymbol{Y}^j = \begin{pmatrix} Y_{1,1}^j & Y_{1,2}^j & \dots & Y_{1,n}^j \\ Y_{2,1}^j & Y_{2,2}^j & \dots & Y_{2,n}^j \end{pmatrix} \qquad j = 1, 2, \dots, m$$

are the measurements in the $j-$th replication of the experiment. In the replicated model the constraints on the parameters are the same as in each particular replication i.e.

$$(\boldsymbol{I}_n \otimes \boldsymbol{B}_0) \operatorname{vec} \boldsymbol{M}_0 + (\boldsymbol{I}_n \otimes \boldsymbol{B}_0 \vdots - \boldsymbol{I}_{2n}) \begin{pmatrix} \operatorname{vec} \delta \boldsymbol{M} \\ \operatorname{vec} \boldsymbol{N} \end{pmatrix} + (\mathbf{1}_n \otimes \boldsymbol{I}_2 \vdots \boldsymbol{M}_0' \otimes \boldsymbol{I}_2) \begin{pmatrix} \boldsymbol{a} \\ \operatorname{vec} \delta \boldsymbol{B} \end{pmatrix} = \mathbf{0}. \tag{15}$$

We shall call the model (14) with constraints (15) as the *replicated two-dimensional comparative calibration model*. This model is also a linear model of measurements with type-II constraints. According to the Appendix 1, the following holds true:

Theorem 2. In the replicated two-dimensional comparative calibration model (14)-(15) the BLUE (best linear unbiased estimators) of the parameters $\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{a}$ and $\delta \boldsymbol{B}$ are given by

$$\operatorname{vec} \widehat{\boldsymbol{M}} = \operatorname{vec} \overline{\boldsymbol{X}} + (\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1})(\operatorname{vec} \overline{\boldsymbol{Y}} - \operatorname{vec} \boldsymbol{B}_0 \overline{\boldsymbol{X}}), \tag{16}$$

$$\operatorname{vec} \widehat{\boldsymbol{N}} = \operatorname{vec} \overline{\boldsymbol{Y}} - (\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1})(\operatorname{vec} \overline{\boldsymbol{Y}} - \operatorname{vec} \boldsymbol{B}_0 \overline{\boldsymbol{X}}), \tag{17}$$

$$\begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec} \widehat{\delta \boldsymbol{B}} \end{pmatrix} = \left(\begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_n' \\ \boldsymbol{M}_0 \end{pmatrix} \otimes \boldsymbol{I}_2 \right) (\operatorname{vec} \overline{\boldsymbol{Y}} - \operatorname{vec} \boldsymbol{B}_0 \overline{\boldsymbol{X}}). \tag{18}$$

where $\overline{\boldsymbol{X}} = \frac{1}{m} \sum_{i=1}^m \boldsymbol{X}^i$, $\overline{\boldsymbol{Y}} = \frac{1}{m} \sum_{i=1}^m \boldsymbol{Y}^i$. The covariance matrix of these estimators are

$$\operatorname{cov} \begin{pmatrix} \operatorname{vec} \widehat{\boldsymbol{M}} \\ \operatorname{vec} \widehat{\boldsymbol{N}} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_X & \mathbf{0}_{2n} \\ \mathbf{0}_{2n} & \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_Y \end{pmatrix} -$$

$$-\frac{1}{m} \begin{pmatrix} \mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1} \boldsymbol{B}_0 \boldsymbol{\Sigma}_X & -\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_X \boldsymbol{B}_0' \boldsymbol{C}^{-1} \boldsymbol{\Sigma}_Y \\ -\mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1} \boldsymbol{B}_0 \boldsymbol{\Sigma}_X & \mathcal{P}_{(\mathbf{1}_n, \boldsymbol{M}_0')} \otimes \boldsymbol{\Sigma}_Y \boldsymbol{C}^{-1} \boldsymbol{\Sigma}_Y \end{pmatrix}, \tag{19}$$

$$\operatorname{cov} \begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec} \widehat{\delta \boldsymbol{B}} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} n & \mathbf{1}_n' \boldsymbol{M}_0' \\ \boldsymbol{M}_0 \mathbf{1}_n & \boldsymbol{M}_0 \boldsymbol{M}_0' \end{pmatrix}^{-1} \otimes \boldsymbol{C}, \tag{20}$$

$$\operatorname{cov}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \operatorname{vec}\widehat{\delta\boldsymbol{B}}\end{pmatrix}, \begin{pmatrix}\operatorname{vec}\widehat{\boldsymbol{M}}\\ \operatorname{vec}\widehat{\boldsymbol{N}}\end{pmatrix}\right) = \frac{1}{m}\left(-\begin{pmatrix} n & \mathbf{1}_n'\boldsymbol{M}_0' \\ \boldsymbol{M}_0\mathbf{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0'\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{1}_n'\\ \boldsymbol{M}_0\end{pmatrix}\otimes\boldsymbol{B}_0\boldsymbol{\Sigma}_X,\right. \tag{21}$$

$$\left.\begin{pmatrix} n & \mathbf{1}_n'\boldsymbol{M}_0' \\ \boldsymbol{M}_0\mathbf{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0'\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{1}_n'\\ \boldsymbol{M}_0\end{pmatrix}\otimes\boldsymbol{\Sigma}_Y\right). \tag{22}$$

Detailed proof can be found in [7], see also Appendix 1.

2.3 MINQUE estimators of the variance components

The covariance matrix of the replicated model (14)-(15) is

$$\boldsymbol{I}_m \otimes \begin{pmatrix}\boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_X & \mathbf{0}_{2n}\\ \mathbf{0}_{2n} & \boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_Y\end{pmatrix}$$

with

$$\boldsymbol{\Sigma}_X = \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}, \quad \boldsymbol{\Sigma}_Y = \begin{pmatrix}\sigma_3^2 & 0\\ 0 & \sigma_4^2\end{pmatrix}$$

which in general depend on unknown parameters – the variance components σ_1^2, σ_2^2, σ_3^2, and σ_4^2. If we have enough measurements, we can estimate the model parameters $\widehat{\boldsymbol{M}}$, $\widehat{\boldsymbol{N}}$, $\widehat{\boldsymbol{a}}$, $\widehat{\delta\boldsymbol{B}}$ simultaneously with the unknown parameters of the covariance matrices σ_i^2, $i = 1,2,3,4$. Here we suggest to estimate this parameters by using the MINQUE estimators (see Appendix 3).

Let us denote

$$\boldsymbol{\Sigma} = \begin{pmatrix}\boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_X & \mathbf{0}_{2n}\\ \mathbf{0}_{2n} & \boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_Y\end{pmatrix},$$

and rewrite the covariance matrices $\boldsymbol{\Sigma}_X$ and $\boldsymbol{\Sigma}_Y$ as

$$\boldsymbol{\Sigma}_X = \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix} = \sigma_1^2\boldsymbol{e}_1\boldsymbol{e}_1' + \sigma_2^2\boldsymbol{e}_2\boldsymbol{e}_2',$$

$$\boldsymbol{\Sigma}_Y = \begin{pmatrix}\sigma_3^2 & 0\\ 0 & \sigma_4^2\end{pmatrix} = \sigma_3^2\boldsymbol{e}_1\boldsymbol{e}_1' + \sigma_4^2\boldsymbol{e}_2\boldsymbol{e}_2',$$

where $\boldsymbol{e}_1 = (1,0)'$ and $\boldsymbol{e}_2 = (0,1)'$. Subsequently, by using the notation

$$\boldsymbol{V}_i = \begin{pmatrix}\boldsymbol{I}_n\otimes\boldsymbol{e}_i\boldsymbol{e}_i' & \mathbf{0}_{2n}\\ \mathbf{0}_{2n} & \mathbf{0}_{3n}\end{pmatrix}, \quad \text{for} \quad i = 1,2,$$

$$\boldsymbol{V}_i = \begin{pmatrix}\mathbf{0}_{2n} & \mathbf{0}_{2n}\\ \mathbf{0}_{2n} & \boldsymbol{I}_n\otimes\boldsymbol{e}_{i-2}\boldsymbol{e}_{i-2}'\end{pmatrix}, \quad \text{for} \quad i = 3,4,$$

we express

$$\boldsymbol{\Sigma} = \sum_{i=1}^{4}\sigma_i^2\boldsymbol{V}_i.$$

Theorem 3. Let $\boldsymbol{\vartheta} = (\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)'$ be the vector of unknown covariance matrix parameters in the replicated two-dimensional comparative calibration model (14),(15). Further, let $\boldsymbol{\vartheta}_0 = (\sigma_{1,0}^2,\ldots,\sigma_{4,0}^2)'$ is the approximate value of the vector parameter $\boldsymbol{\vartheta}$. Analogically, let us denote

$$\boldsymbol{\Sigma}_{X,0} = \begin{pmatrix}\sigma_{1,0}^2 & 0\\ 0 & \sigma_{2,0}^2\end{pmatrix}, \quad \boldsymbol{\Sigma}_{Y,0} = \begin{pmatrix}\sigma_{3,0}^2 & 0\\ 0 & \sigma_{4,0}^2\end{pmatrix},$$

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix}\boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_{X,0} & \mathbf{0}_{2n}\\ \mathbf{0}_{2n} & \boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_{Y,0}\end{pmatrix}.$$

The MINQUE estimator of the unknown vector parameter $\boldsymbol{\vartheta}$ is

$$\widehat{\boldsymbol{\vartheta}} = [(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}]^{-1}\boldsymbol{\kappa}, \tag{23}$$

where $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_4)'$,

$$\kappa_i = \sigma_{i,0}^{-4} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} (\boldsymbol{X}_{i,j}^k - \overline{\boldsymbol{X}}_{i,j})^2 + m(\overline{\boldsymbol{X}}_{i,j} - \widehat{\boldsymbol{M}}_{i,j})^2 \right), \quad i = 1, 2$$

$$\kappa_i = \sigma_{i,0}^{-4} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} (\boldsymbol{Y}_{i-2,j}^k - \overline{\boldsymbol{Y}}_{i-2,j})^2 + m(\overline{\boldsymbol{Y}}_{i-2,j} - \widehat{\boldsymbol{N}}_{i-2,j})^2 \right), \quad i = 3, 4.$$

The covariance matrix of the estimator $\widehat{\boldsymbol{\vartheta}}$ is

$$\mathrm{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0) = 2[(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}]^{-1}, \tag{24}$$

the matrix $\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}}$ is

$$\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} = n \begin{pmatrix} \sigma_{1,0}^{-4} & 0 & 0 & 0 \\ 0 & \sigma_{2,0}^{-4} & 0 & 0 \\ 0 & 0 & \sigma_{3,0}^{-4} & 0 \\ 0 & 0 & 0 & \sigma_{4,0}^{-4} \end{pmatrix}$$

and the matrix $\boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}$ is

$$\boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)} = (n-3) \begin{pmatrix} \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{1,1}^2 & \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{1,2}^2 & \{\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{1,1}^2 & \{\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{2,1}^2 \\ \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{2,1}^2 & \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{2,2}^2 & \{\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{1,2}^2 & \{\boldsymbol{C}_0^{-1}\boldsymbol{B}_0\}_{2,2}^2 \\ \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\}_{1,1}^2 & \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\}_{2,1}^2 & \{\boldsymbol{C}_0^{-1}\}_{1,1}^2 & \{\boldsymbol{C}_0^{-1}\}_{1,2}^2 \\ \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\}_{1,2}^2 & \{\boldsymbol{B}_0'\boldsymbol{C}_0^{-1}\}_{2,2}^2 & \{\boldsymbol{C}_0^{-1}\}_{2,1}^2 & \{\boldsymbol{C}_0^{-1}\}_{2,2}^2 \end{pmatrix}$$

with $\boldsymbol{C}_0 = \boldsymbol{B}_0\boldsymbol{\Sigma}_{X,0}\boldsymbol{B}_0' + \boldsymbol{\Sigma}_{Y,0}$. Detailed proof can be found in [7]. The way to proof the theorem is shown at the end of Appendix 3.

2.4 Iterative estimating procedure

To estimate the desired parameters of the calibration function, we apply results obtained in Section 2.2 and 2.3.

1. We compute the initial (approximative) values of the parameters $\sigma_{1,0}^2, \sigma_{2,0}^2, \sigma_{3,0}^2, \sigma_{4,0}^2$ (as realizations of the following random variables)

$$\sigma_{1,0}^2 = \frac{1}{n(m-1)} \sum_{k=1}^{m} \sum_{j=1}^{n} (\boldsymbol{X}_{1,j}^k - \overline{\boldsymbol{X}}_{1,j})^2,$$

$$\sigma_{2,0}^2 = \frac{1}{n(m-1)} \sum_{k=1}^{m} \sum_{j=1}^{n} (\boldsymbol{X}_{2,j}^k - \overline{\boldsymbol{X}}_{2,j})^2,$$

$$\sigma_{3,0}^2 = \frac{1}{n(m-1)} \sum_{k=1}^{m} \sum_{j=1}^{n} (\boldsymbol{Y}_{1,j}^k - \overline{\boldsymbol{Y}}_{1,j})^2,$$

$$\sigma_{4,0}^2 = \frac{1}{n(m-1)} \sum_{k=1}^{m} \sum_{j=1}^{n} (\boldsymbol{Y}_{2,j}^k - \overline{\boldsymbol{Y}}_{2,j})^2,$$

then we compute the initial values $\boldsymbol{M}_0$ as realizations of the random matrix $\overline{\boldsymbol{X}}$, i.e.

$$\boldsymbol{M}_0 = \overline{\boldsymbol{X}}$$

and the matrix $\boldsymbol{B}_0$ from the realization of the random vector $\operatorname{vec}\widehat{\boldsymbol{B}}_0$, where

$$\operatorname{vec}\hat{\boldsymbol{B}}_0 = (\boldsymbol{0}_{4,2} \vdots \boldsymbol{I}_4)\left[\begin{pmatrix} n & \boldsymbol{1}_n'\overline{\boldsymbol{X}}' \\ \overline{\boldsymbol{X}}\boldsymbol{1}_n & \overline{\boldsymbol{X}}\overline{\boldsymbol{X}}' \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{1}_n' \\ \overline{\boldsymbol{X}} \end{pmatrix} \otimes \boldsymbol{I}_2 \right] \operatorname{vec}\overline{\boldsymbol{Y}}.$$

2. As in Section 2.2, we obtain estimators $\widehat{\boldsymbol{M}}$ and $\widehat{\boldsymbol{N}}$ from (16) and (17) and estimators $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{B}} = \boldsymbol{B}_0 + \widehat{\delta\boldsymbol{B}}$ from (18).
3. We calculate the vector of estimators $\widehat{\boldsymbol{\vartheta}} = (\widehat{\sigma_1^2}, \widehat{\sigma_2^2}, \widehat{\sigma_3^2}, \widehat{\sigma_4^2})'$ from (23).
4. We put the realizations of the estimators $\widehat{\sigma_1^2}, \widehat{\sigma_2^2}, \widehat{\sigma_3^2}, \widehat{\sigma_4^2}$ (i.e. the estimates) as the values $\sigma_{1,0}^2, \sigma_{2,0}^2, \sigma_{3,0}^2, \sigma_{4,0}^2$. Subsequently we put the realizations of $\widehat{\boldsymbol{M}}$ as the initial values $\boldsymbol{M}_0$, the realizations of $\widehat{\boldsymbol{B}}$ as the initial values $\boldsymbol{B}_0$, and return to step 2. We have refined the estimates.

We continue with this iteration process (steps 2, 3, and 4) till the subsequent estimates are sufficiently accurate. According to our opinion 4-7 iteration steps are needed.

2.5 Confidence regions

In the replicated calibration model, besides the estimation of the parameters, it is possible using the method suggested by Kenward and Roger [4] (see Appendix 4) to construct the confidence region for the vector of calibration function parameters and also for the linear function of these parameters. Let us consider linear regression model

$$\begin{pmatrix} \widehat{\boldsymbol{a}} \\ \operatorname{vec}\widehat{\boldsymbol{B}} \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{a} \\ \operatorname{vec}\widehat{\boldsymbol{B}} \end{pmatrix}; \ \frac{1}{m}\begin{pmatrix} n & \boldsymbol{1}_n'\boldsymbol{M}_0' \\ \boldsymbol{M}_0\boldsymbol{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0' \end{pmatrix}^{-1} \otimes (\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)\right). \tag{25}$$

For simplification the notation let us denote

$$\boldsymbol{G} = \frac{1}{m}\begin{pmatrix} n & \boldsymbol{1}_n'\boldsymbol{M}_0' \\ \boldsymbol{M}_0\boldsymbol{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0' \end{pmatrix}^{-1} \otimes (\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y). \tag{26}$$

Since

$$\boldsymbol{\Sigma}_X = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} = \sigma_1^2\boldsymbol{e}_1\boldsymbol{e}_1' + \sigma_2^2\boldsymbol{e}_2\boldsymbol{e}_2', \quad \boldsymbol{\Sigma}_Y = \begin{pmatrix} \sigma_3^2 & 0 \\ 0 & \sigma_4^2 \end{pmatrix} = \sigma_3^2\boldsymbol{e}_1\boldsymbol{e}_1' + \sigma_4^2\boldsymbol{e}_2\boldsymbol{e}_2',$$

we can the covariance matrix $\boldsymbol{G}$ write as

$$\boldsymbol{G} = \sum_{i=1}^{4}\sigma_i^2\boldsymbol{G}_i,$$

where

$$\boldsymbol{D} = \frac{1}{m}\begin{pmatrix} n & \boldsymbol{1}_n'\boldsymbol{M}_0' \\ \boldsymbol{M}_0\boldsymbol{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0' \end{pmatrix}^{-1},$$

$$\boldsymbol{G}_i = \boldsymbol{D} \otimes \boldsymbol{B}_0\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{B}_0' = \boldsymbol{D} \otimes \{\boldsymbol{B}_0\}_{\bullet,i}\{\boldsymbol{B}_0'\}_{\bullet,i} \quad \text{for} \quad i = 1, 2,$$

$$\boldsymbol{G}_i = \boldsymbol{D} \otimes \boldsymbol{e}_{i-2}\boldsymbol{e}_{i-2}' \quad \text{for} \quad i = 3, 4.$$

To follow notation in [4] let us denote

$$\boldsymbol{P}_i = \frac{\partial\boldsymbol{G}^{-1}}{\partial\sigma_i^2} = -\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1},$$

$$\boldsymbol{Q}_{ij} = \frac{\partial\boldsymbol{G}^{-1}}{\partial\sigma_i^2}\boldsymbol{G}\frac{\partial\boldsymbol{G}^{-1}}{\partial\sigma_j^2} = \boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\boldsymbol{G}^{-1},$$

$$\boldsymbol{R}_{ij} = \boldsymbol{G}^{-1} \frac{\partial^2 \boldsymbol{G}}{\partial \sigma_i^2 \partial \sigma_j^2} \boldsymbol{G}^{-1} = \mathbf{0}.$$

As the regression model (25) is a model of direct measurements which covariance matrix is a linear combination of matrices, it holds according to notation in Appendix 4

$$\boldsymbol{\Phi} = \boldsymbol{G} \quad \text{and} \quad \widehat{\boldsymbol{\Phi}} = \widehat{\boldsymbol{G}}.$$

Now let us construct the confidence regions, first for the complete vector of calibration function parameters, then for two particular linear combinations of the parameters. We shall also follow the notation in [4], but in particular subsections we shall use always the same symbols which express the specific situation in the subsection.

2.5.1 Confidence region for the vector of parameters $(\boldsymbol{a}', \operatorname{vec} \boldsymbol{B}')'$

Let us denote

$$\boldsymbol{L} = \boldsymbol{I}_6, \qquad \boldsymbol{\Theta} = \boldsymbol{L}(\boldsymbol{L}'\boldsymbol{G}\boldsymbol{L})^{-1}\boldsymbol{L}' = \boldsymbol{G}^{-1},$$

($\boldsymbol{G}$ is given in (26)) and calculate

$$A_1 = \sum_{i=1}^{4}\sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\} \operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\} = \sum_{i=1}^{4}\sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\} \operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_j\},$$

where

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\} = 3\{\boldsymbol{B}_0'\}_{\bullet,i}(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\{\boldsymbol{B}_0\}_{\bullet,i} \quad \text{for} \ \ i = 1,2,$$

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\} = 3\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{i-2,i-2} \quad \text{for} \ \ i = 3,4,$$

and

$$\boldsymbol{W} = \operatorname{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0)$$

given in (24). Further, we calculate

$$A_2 = \sum_{i=1}^{4}\sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\} = \sum_{i=1}^{4}\sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\},$$

where for $i,j = 1,2$

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\} = 3\{\boldsymbol{B}_0'\}_{\bullet,j}(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\{\boldsymbol{B}_0\}_{\bullet,i}\{\boldsymbol{B}_0'\}_{\bullet,i}(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\{\boldsymbol{B}_0\}_{\bullet,j}$$

for $i = 1,2, \ j = 3,4$

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\} = 3\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{j-2,\bullet}\{\boldsymbol{B}_0\}_{\bullet,i}\{\boldsymbol{B}_0'\}_{\bullet,i}\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{\bullet,j-2}$$

for $i = 3,4, \ j = 1,2$

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\} = 3\{\boldsymbol{B}_0'\}_{\bullet,j}\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{\bullet,i-2}\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{i-2,\bullet}\{\boldsymbol{B}_0\}_{\bullet,j}$$

and for $i,j = 3,4$

$$\operatorname{Tr}\{\boldsymbol{G}^{-1}\boldsymbol{G}_i\boldsymbol{G}^{-1}\boldsymbol{G}_j\} = 3\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{j-2,i-2}\{(\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0' + \boldsymbol{\Sigma}_Y)^{-1}\}_{i-2,j-2}.$$

Utilizing the values A_1, A_2 we calculate successively (see Appendix 4)

$$l = 6, \quad B = \frac{1}{2l}(A_1 + 6A_2) = \frac{1}{12}(A_1 + 6A_2), \quad g = \frac{(l+1)A_1 - (l+4)A_2}{(l+2)A_2} = \frac{7A_1 - 10A_2}{8A_2},$$

$$c_1 = \frac{g}{3l+2(1-g)} = \frac{g}{18+2(1-g)}, \quad c_2 = \frac{l-g}{3l+2(1-g)} = \frac{6-g}{18+2(1-g)},$$

$$c_3 = \frac{l+2-g}{3l+2(1-g)} = \frac{8-g}{18+2(1-g)}, \quad E^* = \left(1-\frac{A_2}{l}\right)^{-1} = \left(1-\frac{A_2}{6}\right)^{-1}$$

$$V^* = \frac{2}{l}\frac{1+c_1B}{(1-c_2B)^2(1-c_3B)} = \frac{1}{3}\frac{1+c_1B}{(1-c_2B)^2(1-c_3B)}, \quad \varrho = \frac{V^*}{2(E^*)^2},$$

$$m^* = 4+\frac{l+2}{l\varrho-1} = 4+\frac{8}{6\varrho-1}, \quad \lambda = \frac{m^*}{E^*(m^*-2)}.$$

Now, according to Appendix 4, we can put down the statistics

$$F = \frac{1}{12}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ vec\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \operatorname{vec}\boldsymbol{B}\end{pmatrix}\right)' \widehat{\boldsymbol{G}}_A^{-1} \left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \operatorname{vec}\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \operatorname{vec}\boldsymbol{B}\end{pmatrix}\right).$$

According to the notation in Appendix 4, $\widehat{\boldsymbol{G}}_A = \widehat{\boldsymbol{\Phi}}_A$. Based on [4], the statistic $F^* = \lambda F$ has approximately the Fisher-Snedecor distribution with 6 and m^* degrees of freedom, i.e.

$$F^* = \lambda F \sim F_{6,m^*}.$$

So, the $(1-\alpha)$ confidence region for the parameters $\boldsymbol{a}, \boldsymbol{B}$ is

$$\left\{\begin{pmatrix}\boldsymbol{a}\\ \operatorname{vec}\boldsymbol{B}\end{pmatrix} : \ \frac{1}{12}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \operatorname{vec}\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \operatorname{vec}\boldsymbol{B}\end{pmatrix}\right)' \widehat{\boldsymbol{G}}_A^{-1} \left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \operatorname{vec}\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \operatorname{vec}\boldsymbol{B}\end{pmatrix}\right) \leq \frac{1}{\lambda}F_{6,m^*}(1-\alpha)\right\}$$

where $F_{6,m^*}(1-\alpha)$ is the $(1-\alpha)$ quantile of F_{6,m^*} distribution.

2.5.2 Confidence intervals for individual calibration function parameters

The confidence interval for each individual parameter of the calibration function is computed as a confidence interval for a special linear combination of all parameters. The vector of all parameters is $(\boldsymbol{a}', (\operatorname{vec}\boldsymbol{B})')'$. It has 6 components. Let us index its components in the natural way, i.e. a_1 is the first parameter, $b_{1,1}$ is the third parameter, $b_{1,2}$ is the fifth parameter, etc. Let us denote ${}_6\boldsymbol{e}_r$ the $r-$th unit 6 dimensional vector, i.e. its $r-$th coordinate is 1 and other 5 coordinates are 0. Let

$$\boldsymbol{L} = {}_6\boldsymbol{e}_r,$$

$$\boldsymbol{\Theta} = \boldsymbol{L}(\boldsymbol{L}'\boldsymbol{G}\boldsymbol{L})^{-1}\boldsymbol{L}' = {}_6\boldsymbol{e}_r({}_6\boldsymbol{e}_r'\boldsymbol{G}{}_6\boldsymbol{e}_r)^{-1}\ {}_6\boldsymbol{e}_r' = \frac{{}_6\boldsymbol{e}_r\ {}_6\boldsymbol{e}_r'}{\{\boldsymbol{G}\}_{r,r}},$$

$\boldsymbol{G}$ is given in (26). So

$$A_1 = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\}\operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\} = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\frac{\{\boldsymbol{G}_i\}_{r,r}\{\boldsymbol{G}_j\}_{r,r}}{\{\boldsymbol{G}\}_{r,r}^2},$$

$$A_2 = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\operatorname{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\} = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\frac{\{\boldsymbol{G}_i\}_{r,r}\{\boldsymbol{G}_j\}_{r,r}}{\{\boldsymbol{G}\}_{r,r}^2},$$

i.e. here $A_1 = A_2$. Further, we calculate (see Appendix 4)

$$l = 1, \quad B = \frac{1}{2l}(A_1+6A_2) = \frac{1}{7}A_1, \quad g = \frac{(l+1)A_1-(l+4)A_2}{(l+2)A_2} = -1,$$

$$c_1 = \frac{g}{3l+2(1-g)} = -\frac{1}{7}, \quad c_2 = \frac{l-g}{3l+2(1-g)} = \frac{2}{7},$$

$$c_3 = \frac{l+2-g}{3l+2(1-g)} = \frac{4}{7}, \quad E^* = \left(1-\frac{A_2}{l}\right)^{-1} = (1-A_1)^{-1},$$

$$V^* = \frac{2}{l}\frac{1+c_1B}{(1-c_2B)^2(1-c_3B)} = \frac{2-A_1}{(1-A_1)^2(1-2A_1)}, \quad \varrho = \frac{V^*}{2(E^*)^2} = \frac{2-A_1}{2(1-2A_1)},$$

$$m^* = 4 + \frac{l+2}{l\varrho-1} = 4 + \frac{3}{\varrho-1} = \frac{4\varrho-1}{\varrho-1} = \frac{2}{A_1}, \quad \lambda = \frac{m^*}{E^*(m^*-2)} = \frac{(1-A_1)(4\varrho-1)}{2\varrho+1} = 1.$$

Now, according to Appendix 4, we can put down the statistics

$$F = \frac{1}{\{\widehat{\boldsymbol{G}}\}_{r,r}}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \text{vec}\,\boldsymbol{B}\end{pmatrix}\right)' {}_6\boldsymbol{e}_r\; {}_6\boldsymbol{e}_r'\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \text{vec}\,\boldsymbol{B}\end{pmatrix}\right).$$

It holds that the statistic $F^* = \lambda F$ has approximately the Fisher-Snedecor distribution with 1 and m^* degrees of freedom, i.e.

$$F \sim F_{1,m^*}.$$

So, the $(1-\alpha)$ confidence interval for the $r-$th parameter $(\boldsymbol{a}', (\text{vec}\,\boldsymbol{B})')$ is

$$\left\{{}_6\boldsymbol{e}_r'\begin{pmatrix}\boldsymbol{a}\\ \text{vec}\,\boldsymbol{B}\end{pmatrix} : \frac{1}{\{\widehat{\boldsymbol{G}}\}_{r,r}}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \text{vec}\,\boldsymbol{B}\end{pmatrix}\right)' {}_6\boldsymbol{e}_r\; {}_6\boldsymbol{e}_r'\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \text{vec}\,\boldsymbol{B}\end{pmatrix}\right) \le F_{1,m^*}(1-\alpha)\right\},$$

or

$$\left\langle {}_6\boldsymbol{e}_r'\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \sqrt{\{\widehat{\boldsymbol{G}}\}_{r,r}F_{1,m^*}(1-\alpha)}, {}_6\boldsymbol{e}_r'\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \text{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} + \sqrt{\{\widehat{\boldsymbol{G}}\}_{r,r}F_{1,m^*}(1-\alpha)}\right\rangle.$$

2.5.3 Confidence intervals for two important linear functions of parameters

Let us denote μ_1 and μ_2 the errorless (true, ideal) values measured with the measuring device $\mathcal{X}$ (in units of the measuring device $\mathcal{X}$). Let

$$\boldsymbol{L}^{(1)} = (1, 0, \mu_1, 0, \mu_2, 0)',$$

$$\boldsymbol{\Theta}^{(1)} = \boldsymbol{L}^{(1)}\left((\boldsymbol{L}^{(1)})'\boldsymbol{G}\boldsymbol{L}^{(1)}\right)^{-1}(\boldsymbol{L}^{(1)})'$$

($\boldsymbol{G}$ is given in (26)). First we calculate

$$\left((\boldsymbol{L}^{(1)})'\boldsymbol{G}\boldsymbol{L}^{(1)}\right)^{-1}$$

and then

$$A_1^{(1)} = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\,\text{Tr}\{\boldsymbol{\Theta}^{(1)}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\}\,\text{Tr}\{\boldsymbol{\Theta}^{(1)}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\},$$

$$A_2^{(1)} = \sum_{i=1}^{4}\sum_{j=1}^{4}\{\boldsymbol{W}\}_{i,j}\,\text{Tr}\{\boldsymbol{\Theta}^{(1)}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\boldsymbol{\Theta}^{(1)}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\}.$$

Further, we calculate (according to Appendix A)

$$l^{(1)} = 1,\; B^{(1)} = \frac{1}{2l^{(1)}}(A_1^{(1)} + 6A_2^{(1)}),\; g^{(1)} = \frac{(l^{(1)}+1)A_1^{(1)} - (l^{(1)}+4)A_2^{(1)}}{(l^{(1)}+2)A_2^{(1)}},\; c_1^{(1)} = \frac{g^{(1)}}{3l^{(1)}+2(1-g^{(1)})},$$

$$c_2^{(1)} = \frac{l-g^{(1)}}{3l^{(1)}+2(1-g^{(1)})}, \quad c_3^{(1)} = \frac{l^{(1)}+2-g^{(1)}}{3l^{(1)}+2(1-g^{(1)})}, \quad E^{*(1)} = \left(1-\frac{A_2^{(1)}}{l^{(1)}}\right)^{-1},$$

$$V^{*(1)} = \frac{2}{l^{(1)}} \frac{1 + c_1^{(1)} B^{(1)}}{(1 - c_2^{(1)} B^{(1)})^2 (1 - c_3^{(1)} B^{(1)})}, \quad \varrho = \frac{V^{*(1)}}{2 {E^{*(1)}}^2}, \quad m^{*(1)} = 4 + \frac{l^{(1)} + 2}{l^{(1)} \varrho^{(1)} - 1},$$

$$\lambda^{(1)} = \frac{m^{*(1)}}{E^{*(1)} (m^{*(1)} - 2)}.$$

According to Appendix 4 we obtain the statistics

$$F = \frac{1}{(\boldsymbol{L}^{(1)})' \widehat{\boldsymbol{G}} \boldsymbol{L}^{(1)}} \left(\widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - (a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2) \right)'$$

$$\left(\widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - (a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2) \right).$$

It holds that the statistic $F^* = \lambda^{(1)} F$ has approximately the Fisher-Snedecor distribution with 1 and $m^{*(1)}$ degrees of freedom, i.e.,

$$F \sim \frac{1}{\lambda^{(1)}} F_{1, m^{*(1)}}.$$

The $(1 - \alpha)$ confidence interval for $\nu_1 = a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2$ is

$$\Bigg\{ \nu_1 = a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2 : \frac{1}{\left((\boldsymbol{L}^{(1)})' \widehat{\boldsymbol{G}} \boldsymbol{L}^{(1)} \right)} \left(\widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - (a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2) \right)'$$

$$\left(\widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - (a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2) \right) \leq \frac{1}{\lambda^{(1)}} F_{1, m^{*(1)}}(1 - \alpha) \Bigg\},$$

or

$$\Bigg\langle \widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - \sqrt{\frac{(\boldsymbol{L}^{(1)})' \widehat{\boldsymbol{G}} \boldsymbol{L}^{(1)}}{\lambda^{(1)}} F_{1, m^{*(1)}}(1 - \alpha)},$$

$$\widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 + \sqrt{\frac{(\boldsymbol{L}^{(1)})' \widehat{\boldsymbol{G}} \boldsymbol{L}^{(1)}}{\lambda^{(1)}} F_{1, m^{*(1)}}(1 - \alpha)} \Bigg\rangle.$$

We only note that $\boldsymbol{L}^{(1)}$ depends on μ_1, μ_2, i.e. $\boldsymbol{L}^{(1)} = \boldsymbol{L}^{(1)}(\mu_1, \mu_2)$, $m^{*(1)}$ depends on μ_1, μ_2, i.e. $m^{*(1)} = m^{*(1)}(\mu_1, \mu_2)$ and also $\lambda^{(1)}$ depends on μ_1, μ_2, i.e. $\lambda^{(1)} = \lambda^{(1)}(\mu_1, \mu_2)$.

Let now

$$\boldsymbol{L}^{(2)} = (0, 1, 0, \mu_1, 0, \mu_2)',$$

$$\boldsymbol{\Theta}^{(2)} = \boldsymbol{L}^{(2)} \left((\boldsymbol{L}^{(2)})' \boldsymbol{G} \boldsymbol{L}^{(2)} \right)^{-1} (\boldsymbol{L}^{(2)})'$$

($\boldsymbol{G}$ is given in (26)). First we calculate

$$\left((\boldsymbol{L}^{(2)})' \boldsymbol{G} \boldsymbol{L}^{(2)} \right)^{-1}$$

and then

$$A_1^{(2)} = \sum_{i=1}^{4} \sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta}^{(2)} \boldsymbol{G} \boldsymbol{P}_i \boldsymbol{G}\} \operatorname{Tr}\{\boldsymbol{\Theta}^{(2)} \boldsymbol{G} \boldsymbol{P}_j \boldsymbol{G}\},$$

$$A_2^{(2)} = \sum_{i=1}^{4} \sum_{j=1}^{4} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta}^{(2)} \boldsymbol{G} \boldsymbol{P}_i \boldsymbol{G} \boldsymbol{\Theta}^{(2)} \boldsymbol{G} \boldsymbol{P}_j \boldsymbol{G}\}.$$

Further we calculate (according to Appendix 4)

$$l^{(2)} = 1,\ B^{(2)} = \frac{1}{2l^{(2)}}(A_1^{(2)} + 6A_2^{(2)}),\ g^{(1)} = \frac{(l^{(2)}+1)A_1^{(2)} - (l^{(2)}+4)A_2^{(2)}}{(l^{(2)}+2)A_2^{(2)}},\ c_1^{(2)} = \frac{g^{(2)}}{3l^{(2)} + 2(1-g^{(2)})},$$

$$c_2^{(2)} = \frac{l - g^{(2)}}{3l^{(2)} + 2(1-g^{(2)})},\quad c_3^{(2)} = \frac{l^{(2)} + 2 - g^{(2)}}{3l^{(2)} + 2(1-g^{(2)})},\quad E^{*(2)} = \left(1 - \frac{A_2^{(2)}}{l^{(2)}}\right)^{-1},$$

$$V^{*(2)} = \frac{2}{l^{(2)}} \frac{1 + c_1^{(2)}B^{(2)}}{(1 - c_2^{(2)}B^{(2)})^2(1 - c_3^{(2)}B^{(2)})},\quad \varrho = \frac{V^{*(2)}}{2{E^{*(2)}}^2},\quad m^{*(2)} = 4 + \frac{l^{(2)}+2}{l^{(2)}\varrho^{(2)} - 1},$$

$$\lambda^{(2)} = \frac{m^{*(2)}}{E^{*(2)}(m^{*(2)} - 2)}.$$

Again, according to Appendix 4, we obtain the statistics

$$F = \frac{1}{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(1)}} \left(\widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - (a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2)\right)'$$

$$\left(\widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - (a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2)\right).$$

It holds that the statistic $F^* = \lambda^{(2)}F$ has approximately the Fisher-Snedecor distribution with 1 and $m^{*(2)}$ degrees of freedom, i.e.,

$$F \sim \frac{1}{\lambda^{(2)}} F_{1,m^{*(2)}}.$$

The $(1-\alpha)$ confidence interval for $\nu_2 = a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2$ is

$$\Bigg\{\nu_2 = a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2 : \frac{1}{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(2)}} \left(\widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - (a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2)\right)'$$

$$\left(\widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - (a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2)\right) \leq \frac{1}{\lambda^{(2)}} F_{1,m^{*(2)}}(1-\alpha)\Bigg\},$$

or

$$\Bigg\langle \widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - \sqrt{\frac{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(2)}}{\lambda^{(2)}} F_{1,m^{*(2)}}(1-\alpha)},$$

$$\widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{21,2}\mu_2 + \sqrt{\frac{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(2)}}{\lambda^{(2)}} F_{1,m^{*(2)}}(1-\alpha)}\Bigg\rangle.$$

We only note that $\boldsymbol{L}^{(2)}$ again depends on μ_1, μ_2, i.e. $\boldsymbol{L}^{(2)} = \boldsymbol{L}^{(2)}(\mu_1, \mu_2)$, $m^{*(2)}$ depends on μ_1, μ_2, i.e. $m^{*(2)} = m^{*(2)}(\mu_1, \mu_2)$ and also $\lambda^{(2)}$ depends on μ_1, μ_2, i.e. $\lambda^{(2)} = \lambda^{(2)}(\mu_1, \mu_2)$.

2.5.4 Confidence region for $(\nu_1(\mu_1, \mu_2), \nu_2(\mu_1, \mu_2))'$

$\boldsymbol{\mu} = (\mu_1, \mu_2)'$ are as former the errorless (true, ideal) values measured with the measuring device $\mathcal{X}$ (in units of the measuring device $\mathcal{X}$). The vector $\boldsymbol{a} + \boldsymbol{B}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}\nu_1(\mu_1, \mu_2)\\ \nu_2(\mu_1, \mu_2)\end{pmatrix}$ respond to the calibration function values. Let

$$\boldsymbol{L} = \boldsymbol{\mu}^* \otimes \boldsymbol{I}_2,\quad \boldsymbol{\mu}^* = \begin{pmatrix}1\\ \mu_1\\ \mu_2\end{pmatrix}.$$

Then it holds

$$\boldsymbol{L}'\begin{pmatrix}\boldsymbol{a}\\ \mathrm{vec}\,\boldsymbol{B}\end{pmatrix} = \boldsymbol{a}+\boldsymbol{B}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \boldsymbol{a}+\boldsymbol{B}\boldsymbol{\mu}.$$

Further,

$$\boldsymbol{\Theta} = \boldsymbol{L}(\boldsymbol{L}'\boldsymbol{G}\boldsymbol{L})^{-1}\boldsymbol{L}' = [\boldsymbol{\mu}^*\otimes\boldsymbol{I}_2]([(\boldsymbol{\mu}^*)'\otimes\boldsymbol{I}_2][\boldsymbol{D}\otimes\boldsymbol{C}][\boldsymbol{\mu}^*\otimes\boldsymbol{I}_2])^{-1}[(\boldsymbol{\mu}^*)'\otimes\boldsymbol{I}_2] =$$

$$= [\boldsymbol{\mu}^*\otimes\boldsymbol{I}_2][((\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*)^{-1}\otimes\boldsymbol{C}^{-1}][(\boldsymbol{\mu}^*)'\otimes\boldsymbol{I}_2] = \frac{\boldsymbol{\mu}^*(\boldsymbol{\mu}^*)'}{(\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*}\otimes\boldsymbol{C}^{-1},$$

where $\boldsymbol{G}=\boldsymbol{D}\otimes\boldsymbol{C}$, $\boldsymbol{C}=\boldsymbol{B}_0\boldsymbol{\Sigma}_X\boldsymbol{B}_0'+\boldsymbol{\Sigma}_Y$ and $\boldsymbol{D}=\frac{1}{m}\begin{pmatrix}n & \boldsymbol{1}_n'\boldsymbol{M}_0'\\ \boldsymbol{M}_0\boldsymbol{1}_n & \boldsymbol{M}_0\boldsymbol{M}_0'\end{pmatrix}^{-1}$. So,

$$A_1 = \sum_{i=1}^4\sum_{j=1}^4\{\boldsymbol{W}\}_{i,j}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\} = \sum_{i=1}^4\sum_{j=1}^4\{\boldsymbol{W}\}_{i,j}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_j\},$$

with

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\} = \mathrm{Tr}\left\{\left[\frac{\boldsymbol{\mu}^*(\boldsymbol{\mu}^*)'}{(\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*}\otimes\boldsymbol{C}^{-1}\right][\boldsymbol{D}\otimes\{\boldsymbol{B}_0\}_{\bullet,i}\{\boldsymbol{B}_0'\}_{\bullet,i}]\right\} = \{\boldsymbol{B}_0'\}_{\bullet,i}\boldsymbol{C}^{-1}\{\boldsymbol{B}_0\}_{\bullet,i},\quad i=1,2,$$

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\} = \mathrm{Tr}\left\{\left[\frac{\boldsymbol{\mu}^*(\boldsymbol{\mu}^*)'}{(\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*}\otimes\boldsymbol{C}^{-1}\right][\boldsymbol{D}\otimes\ {}_2\boldsymbol{e}_{i-2}\ {}_2\boldsymbol{e}_{i-2}']\right\} = \{\boldsymbol{C}^{-1}\}_{i-2,i-2},\quad i=3,4.$$

Furthermore let us compute

$$A_2 = \sum_{i=1}^4\sum_{j=1}^4\{\boldsymbol{W}\}_{i,j}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_i\boldsymbol{G}\boldsymbol{\Theta}\boldsymbol{G}\boldsymbol{P}_j\boldsymbol{G}\} = \sum_{i=1}^4\sum_{j=1}^4\{\boldsymbol{W}\}_{i,j}\,\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\boldsymbol{\Theta}\boldsymbol{G}_j\}.$$

After tedious computations we obtain

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\boldsymbol{\Theta}\boldsymbol{G}_j\} = [\{\boldsymbol{B}_0'\}_{\bullet,j}\boldsymbol{C}^{-1}\{\boldsymbol{B}_0\}_{\bullet,i}]^2,\quad i,j=1,2,$$

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\boldsymbol{\Theta}\boldsymbol{G}_j\} = [\{\boldsymbol{C}^{-1}\}_{j-2,\bullet}\{\boldsymbol{B}_0\}_{\bullet,i}]^2,\quad i,j=3,4,$$

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\boldsymbol{\Theta}\boldsymbol{G}_j\} = [\{\boldsymbol{B}_0'\}_{\bullet,j}\{\boldsymbol{C}^{-1}\}_{\bullet,i-2}]^2,\quad i=3,4,\quad j=1,2,$$

$$\mathrm{Tr}\{\boldsymbol{\Theta}\boldsymbol{G}_i\boldsymbol{\Theta}\boldsymbol{G}_j\} = [\{\boldsymbol{C}^{-1}\}_{j-2,i-2}]^2,\quad i,j=3,4.$$

Using the values of A_1 and A_2 according to Appendix 4 we obtain

$$l=2,\quad B=\frac{1}{2l}(A_1+6A_2)=\frac{1}{4}(A_1+6A_2),\quad g=\frac{(l+1)A_1-(l+4)A_2}{(l+2)A_2}=\frac{3}{4}\frac{A_1-2A_2}{A_2},$$

$$c_1=\frac{g}{3l+2(1-g)}=\frac{g}{8-2g},\quad c_2=\frac{l-g}{3l+2(1-g)}=\frac{2-g}{8-2g},$$

$$c_3=\frac{l+2-g}{3l+2(1-g)}=\frac{4-g}{8-2g},\quad E^*=\left(1-\frac{A_2}{l}\right)^{-1}=2(2-A_1)^{-1},$$

$$V^*=\frac{2}{l}\frac{1+c_1B}{(1-c_2B)^2(1-c_3B)}=\frac{1+c_1B}{(1-c_2B)^2(1-c_3B)},\quad \varrho=\frac{V^*}{2(E^*)^2},$$

$$m^*=4+\frac{l+2}{l\varrho-1}=4+\frac{4}{2\varrho-1}=\frac{8\varrho}{2\varrho-1},\quad \lambda=\frac{m^*}{E^*(m^*-2)}=\frac{3-A_2}{9}\,\frac{12\varrho+1}{2\varrho+1}.$$

Now, according to Appendix 4, we can put down the statistics

$$F = \frac{1}{2(\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*}\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \mathrm{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \mathrm{vec}\,\boldsymbol{B}\end{pmatrix}\right)'\left(\boldsymbol{\mu}^*(\boldsymbol{\mu}^*)'\otimes\widehat{\boldsymbol{C}}^{-1}\right)\left(\begin{pmatrix}\widehat{\boldsymbol{a}}\\ \mathrm{vec}\,\widehat{\boldsymbol{B}}\end{pmatrix} - \begin{pmatrix}\boldsymbol{a}\\ \mathrm{vec}\,\boldsymbol{B}\end{pmatrix}\right).$$

It holds that the statistic $F^* = \lambda F$ has approximately the Fisher-Snedecor distribution with 2 and m^* degrees of freedom, i.e.

$$\lambda F \sim F_{2,m^*}.$$

So, the $(1-\alpha)$ confidence region for $\boldsymbol{a} + \boldsymbol{B}\boldsymbol{\mu}^*$ is

$$\left\{\begin{pmatrix}\nu_1(\mu_1,\mu_2)\\ \nu_2(\mu_1,\mu_2)\end{pmatrix} = \boldsymbol{a}+\boldsymbol{B}\boldsymbol{\mu}:\quad \frac{1}{2(\boldsymbol{\mu}^*)'\boldsymbol{D}\boldsymbol{\mu}^*}\left(\widehat{\boldsymbol{a}}+\widehat{\boldsymbol{B}}\boldsymbol{\mu}-(\boldsymbol{a}+\boldsymbol{B}\boldsymbol{\mu})\right)'\widehat{\boldsymbol{C}}^{-1}\left(\widehat{\boldsymbol{a}}+\widehat{\boldsymbol{B}}\boldsymbol{\mu}-(\boldsymbol{a}+\boldsymbol{B}\boldsymbol{\mu})\right) \le \frac{1}{\lambda}F_{2,m^*}(1-\alpha)\right\}.$$

2.6 Measuring with the calibrated device

Let us assume that by the measuring device $\mathcal{X}$ (the less precise measuring device, the measuring device to be calibrated) we have recorded an errorless couple of values $\boldsymbol{\mu} = (\mu_1, \mu_2)'$. We want to determine (using the Kenward and Roger's method) the (approximate) $(1-\alpha)$ confidence region for the value $\boldsymbol{\nu}_{\boldsymbol{\mu}}$ i.e. the $(1-\alpha)$ confidence region for the errorless recorded value $\boldsymbol{a}+\boldsymbol{B}\boldsymbol{\mu}$ measured by the standard $\mathcal{Y}$. The desired $(1-\alpha)$ confidence region is given in Section 2.5.4.

Now let us determine the confidence region for $\boldsymbol{\nu}_{\boldsymbol{\mu}}$ when (errorless, true) couple $\boldsymbol{\mu}$ is measured by the measuring device $\mathcal{X}$, but the realization of the measurement (the registered couple of values, the evidence couple) is $\boldsymbol{x} = (x_1, x_2)'$. It means we have realized the measurement $\boldsymbol{X} = \begin{pmatrix}X_1\\ X_2\end{pmatrix} \sim N\left(\boldsymbol{\mu}, \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\right)$. The measurements X_1 and X_2 are independent.

It is well known that if $X_1 \sim N(\mu_1, \sigma_1^2)$ and S_1^2 is an estimator of σ_1^2 for which it holds $\frac{w_1}{\sigma_1^2}S_1^2 \sim \chi^2_{w_1}$ ($\chi^2_{w_1}$ is the χ^2 distribution with w_1 degrees of freedom), while X_1 and S_1^2 are independent, then

$$\frac{X_1-\mu_1}{S_1} \sim t_{w_1}$$

(t_{w_1} is the Student's t distribution with w_1 degrees of freedom). The dispersion of S_1^2, $\mathcal{D}(S_1^2) = \frac{2\sigma_1^4}{w_1}$ and the degrees of freedom $w_1 = \frac{2\sigma_1^4}{\mathcal{D}(S_1^2)}$. If we substitute S_1^2 by the MINQUE estimator $\widehat{\sigma}_1^2$, then the degrees of freedom are given approximately as

$$w_1 \doteq \frac{2\widehat{\sigma}_1^4}{\left\{\mathrm{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0)\right\}_{1,1}}$$

where $\mathrm{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0)$ is given in (24).

The $(1-\gamma)$ confidence interval for μ_1 is

$$U_1^{(1-\gamma)} = \langle x_1 - \widehat{\sigma}_1 t_{w_1}(1-\gamma/2), x_1 + \widehat{\sigma}_1 t_{w_1}(1-\gamma/2)\rangle,$$

where $t_{w_1}(1-\gamma/2)$ is the $(1-\gamma/2)$ quantil of the t_{w_1} distribution. Analogically the $(1-\gamma)$ confidence interval for μ_2 is

$$U_2^{(1-\gamma)} = \langle x_1 - \widehat{\sigma}_2 t_{w_2}(1-\gamma/2), x_2 + \widehat{\sigma}_2 t_{w_2}(1-\gamma/2)\rangle.$$

Using Bonferroni inequality we easy obtain that $U_1^{(1-\gamma)} \cap U_2^{(1-\gamma)}$ is at least $(1-2\gamma)$ confidence region for (μ_1, μ_2), i.e.

$$P((\mu_1,\mu_2) \in \langle x_1 - \widehat{\sigma}_1 t_{w_1}(1-\gamma/4), x_1 + \widehat{\sigma}_1 t_{w_1}(1-\gamma/4)\rangle \times \langle x_1 - \widehat{\sigma}_2 t_{w_2}(1-\gamma/4), x_2 + \widehat{\sigma}_2 t_{w_2}(1-\gamma/4)\rangle$$

$$= U_1^{(1-\gamma/2)} \times U_2^{(1-\gamma/2)}) \geq 1-\gamma.$$

If we denote

$$d_1 = \inf_{(\mu_1,\mu_2)\in U_1^{(1-\gamma/2)}\times U_2^{(1-\gamma/2)}} \left\{ \widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 - \sqrt{\frac{(\boldsymbol{L}^{(1)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(1)}}{\lambda^{(1)}} F_{1,m^{*(1)})}(1-\alpha/2)} \right\},$$

$$h_1 = \sup_{(\mu_1,\mu_2)\in U_1^{(1-\gamma/2)}\times U_2^{(1-\gamma/2)}} \left\{ \widehat{a}_1 + \widehat{b}_{1,1}\mu_1 + \widehat{b}_{1,2}\mu_2 + \sqrt{\frac{(\boldsymbol{L}^{(1)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(1)}}{\lambda^{(1)}(\mu_1,\mu_2)} F_{1,m^{*(1)}}(1-\alpha/2)} \right\},$$

where $L^{(1)} = L^{(1)}(\mu_1,\mu_2),\ \lambda^{(1)} = \lambda^{(1)}(\mu_1,\mu_2)$ and $m^{*(1)} = m^{*(1)}(\mu_1,\mu_2)$ are defined in Section 2.5.3 and

$$d_2 = \inf_{(\mu_1,\mu_2)\in U_1^{(1-\gamma/2)}\times U_2^{(1-\gamma/2)}} \left\{ \widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 - \sqrt{\frac{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(2)}}{\lambda^{(2)}} F_{1,m^{*(2)}}(1-\alpha/2)} \right\},$$

$$h_2 = \sup_{(\mu_1,\mu_2)\in U_1^{(1-\gamma/2)}\times U_2^{(1-\gamma/2)}} \left\{ \widehat{a}_2 + \widehat{b}_{2,1}\mu_1 + \widehat{b}_{2,2}\mu_2 + \sqrt{\frac{(\boldsymbol{L}^{(2)})'\widehat{\boldsymbol{G}}\boldsymbol{L}^{(2)}}{\lambda^{(1)}} F_{1,m^{*(2)}}(1-\alpha/2)} \right\},$$

where $L^{(2)} = L^{(2)}(\mu_1,\mu_2),\ \lambda^{(2)} = \lambda^{(2)}(\mu_1^*,\mu_2^*)$ and $m^{*(2)} = m^{*(2)}(\mu_1,\mu_2)$ are defined in Section 2.5.3, then

$$(d_1,h_1)\times(d_2,h_2)$$

is the wanted at least $(1-\alpha-\gamma)$ confidence region for $\boldsymbol{\nu}_\mu$. Practically is suggested to calculate the above supremas and infimas on a sufficiently dense grid on the set $(U_1^{(1-\gamma/2)} \times U_2^{(1-\gamma/2)})$.

The suggested estimator of $\nu_1(\mu_1,\mu_2) = a_1 + b_{1,1}\mu_1 + b_{1,2}\mu_2$ is $\dfrac{d_1+h_1}{2}$ and

the suggested estimator of $\nu_2(\mu_1,\mu_2) = a_2 + b_{2,1}\mu_1 + b_{2,2}\mu_2$ is $\dfrac{d_2+h_2}{2}$.

3 Conclusions

We derived the two-dimensional comparative linear calibration model. This model is an errors-in-variables model and after linearization could be represented as a linear regression model with linear constraints on the parameters. It is supposed that the measurements are independent and normally distributed. If the uncertainties of measuremets are not known, but constant for each part of the measured couples, derived were the MINQUE estimators of these uncertainties. The optimal linear estimators (BLUEs) of the model parameters are shown. The approximate $(1-\alpha)$ confidence regions are derived for the whole vector of parameters and also for any linear function of the parameters, especially for each separate parameter using results obtained by Kenward and Roger [4]. Further research will be focused on considering also Type B unceraintties, dependency between measurements, and also other types of calibration function.

Appendix 1. Linear measurement model with constraints on the parameters

Sometimes called as regression model with type-II constraints. According to [2] many experiments lead to this type of model, e.g. calibration curve, balance equations in chemistry, estimation of the position of the isobestic point in chemistry, estimation of parameters of trajectory on the Earth's surface on the basic of measured position of several points on this trajectory, etc.

Let $\boldsymbol{\beta}_1$ is a k_1 dimensional real vector $\boldsymbol{\beta}_2$ is a k_2 dimensional real vector, $\boldsymbol{b}$ is a q dimensional real vector, $\boldsymbol{B}_1$ is a $q\times k_1$ real matrix and $\boldsymbol{B}_2$ is a $q\times k_2$ real matrix with rank $r(\boldsymbol{B}_1,\boldsymbol{B}_2) = q \leq k_1+k_2$, $r(\boldsymbol{B}_2) = k_2 \leq q$. Further let $\boldsymbol{X}$ be a real $n\times k_1$ real matrix with $r(\boldsymbol{X}) = k_1 \leq n$, $\boldsymbol{\Sigma}$ is a positive definite $n\times n$ real matrix and $\boldsymbol{Y}$ is an n dimensional random vector with the mean $\boldsymbol{X\beta}_1$ and covariance matric $\boldsymbol{\Sigma}$. Then the triplet $(\boldsymbol{Y}, \boldsymbol{X\beta}_1, \boldsymbol{\Sigma})$ with constraints $\boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2=\boldsymbol{0}$ is called a *nonreplicated (regular) regression model with type-II constraints* and written as

$$\boldsymbol{Y} \sim (\boldsymbol{X\beta}_1, \boldsymbol{\Sigma}), \quad \boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2=\boldsymbol{0}.$$

Theorem 4 Let us have a (regular) regression model with type-II constraints. Let

$$\begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})\boldsymbol{B}_1' & \boldsymbol{B}_2 \\ \boldsymbol{B}_2' & \boldsymbol{0} \end{pmatrix}^{-1}$$

Then the BLUE (best linear unbiased estimator) of parameters $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ is

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{pmatrix} = -\begin{pmatrix} (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\boldsymbol{Q}_{11} \\ \boldsymbol{Q}_{21} \end{pmatrix}\boldsymbol{b} + \begin{pmatrix} \boldsymbol{I} - (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1 \\ -\boldsymbol{Q}_{21}\boldsymbol{B}_1 \end{pmatrix}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}.$$

The covariance matrix of the estimator is

$$\operatorname{cov}\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{pmatrix} = \begin{pmatrix} \operatorname{cov}(\widehat{\boldsymbol{\beta}}_1) & \operatorname{cov}(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2) \\ \operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1) & \operatorname{cov}(\widehat{\boldsymbol{\beta}}_2) \end{pmatrix},$$

where

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_1) = (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1} - (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1},$$

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2) = -\boldsymbol{Q}_{22},$$

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1) = -\boldsymbol{Q}_{21}\boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1} = (\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1))'.$$

Proof is given in [5], [6].

Now we shall introduce the replicated measurement model. Let $\boldsymbol{Y}_1 \sim (\boldsymbol{X\beta}_1, \boldsymbol{\Sigma})$, $\boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2=\boldsymbol{0}, ..., \boldsymbol{Y}_m \sim (\boldsymbol{X\beta}_1, \boldsymbol{\Sigma})$, $\boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2=\boldsymbol{0}$ are (regular) regression models with type-II constraints. Let $\operatorname{cov}(\boldsymbol{Y}_i, \boldsymbol{Y}_j) = \boldsymbol{0}$ for $i \neq j$. Let us denote $\underline{\boldsymbol{Y}} = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')'$. Then $\underline{\boldsymbol{Y}} \sim [(\boldsymbol{1}_m \otimes \boldsymbol{X}), \boldsymbol{I}_m \otimes \boldsymbol{\Sigma}]$ with constraints $\boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2=\boldsymbol{0}$ is called a *replicated (regular) regression model with type-II constraints*.

Theorem 5 Let us have a replicated (regular) regression model with type-II constraints. Let us denote $\overline{\boldsymbol{Y}} = \frac{1}{m}\sum_{j=1}^{m}\boldsymbol{Y}_j$ and

$$\begin{pmatrix} \widetilde{\boldsymbol{Q}}_{11} & \widetilde{\boldsymbol{Q}}_{12} \\ \widetilde{\boldsymbol{Q}}_{21} & \widetilde{\boldsymbol{Q}}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}\boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})\boldsymbol{B}_1' & \boldsymbol{B}_2 \\ \boldsymbol{B}_2' & \boldsymbol{0} \end{pmatrix}^{-1}$$

Then the BLUE (best linear unbiased estimator) of parameters $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ is

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{pmatrix} = -\begin{pmatrix} \frac{1}{m}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11} \\ \widetilde{\boldsymbol{Q}}_{21} \end{pmatrix}\boldsymbol{b} + \begin{pmatrix} \boldsymbol{I} - \frac{1}{m}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1 \\ -\widetilde{\boldsymbol{Q}}_{21}\boldsymbol{B}_1 \end{pmatrix}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{Y}}.$$

The covariance matrix of the estimator is

$$\operatorname{cov}\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{pmatrix} = \begin{pmatrix} \operatorname{cov}(\widehat{\boldsymbol{\beta}}_1) & \operatorname{cov}(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2) \\ \operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1) & \operatorname{cov}(\widehat{\boldsymbol{\beta}}_2) \end{pmatrix},$$

where

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_1) = \frac{1}{m}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1} - \frac{1}{m^2}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1},$$

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2) = -\widetilde{\boldsymbol{Q}}_{22},$$

$$\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1) = -\tfrac{1}{m}\widetilde{\boldsymbol{Q}}_{21}\boldsymbol{B}_1(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1} = (\operatorname{cov}(\widehat{\boldsymbol{\beta}}_2, \widehat{\boldsymbol{\beta}}_1))'.$$

The proof follows from Theorem 4 if we substitute $\boldsymbol{X} \to \mathbf{1}_m \otimes \boldsymbol{X}$, $\boldsymbol{\Sigma} \to \boldsymbol{I}_m \otimes \boldsymbol{\Sigma}$ and $\boldsymbol{Y} \to \underline{\boldsymbol{Y}}$.

Appendix 2. Equivalence of model with constraints on the parameters and the transformed model without constraints on the parameters

We can use the MINQUE estimators of the covariance matrix parameters only in replicated model without constraints on the parameters. That is why it is necessary to transform the linear model with constraints on the parameters to an equivalent model without constraints on the parameters.

Let us have two models which contains the same vector of unknown parameters. This two models we call *equivalent models* with connection to the unknown parameters and estimation method if the estimates of the unknown parameters, using the chosen method, are in both models the same.

Let us consider the regular regression model with type-II constraints (see Appendix 1)

$$\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \qquad \boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \mathbf{0}.$$

The matrix $\boldsymbol{X}$ is here the matrix $\boldsymbol{I}_{k_1}$. The vector $\boldsymbol{b}$ let us split into two parts, namely a $q - k_2$ dimensional vector $\boldsymbol{b}_1$ and a k_2 dimensuonal vector $\boldsymbol{b}_2$. Matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ let us split into two submatrices, namely $\boldsymbol{B}_1' = (\boldsymbol{B}_{11}', \boldsymbol{B}_{12}')$, where $\boldsymbol{B}_{12}$ is a $k_1 \times k_2$ matrix and $\boldsymbol{B}_2$ is a $k_2 \times k_2$ matrix. Moreover let us (without lose of generality) suppose, that $\boldsymbol{B}_{22}$ is regular. The constraints $\boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \mathbf{0}$ we rewrite as

$$\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{12} \end{pmatrix} \boldsymbol{\beta}_1 + \begin{pmatrix} \boldsymbol{B}_{21} \\ \boldsymbol{B}_{22} \end{pmatrix} \boldsymbol{\beta}_2 = \mathbf{0}.$$

So we obtain a system of two linear equations

$$\boldsymbol{b}_1 + \boldsymbol{B}_{11}\boldsymbol{\beta}_1 + \boldsymbol{B}_{21}\boldsymbol{\beta}_2 = \mathbf{0},$$

$$\boldsymbol{b}_2 + \boldsymbol{B}_{12}\boldsymbol{\beta}_1 + \boldsymbol{B}_{22}\boldsymbol{\beta}_2 = \mathbf{0}.$$

From the second equation is

$$\boldsymbol{\beta}_2 = -\boldsymbol{B}_{22}^{-1}\boldsymbol{b}_2 - \boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{12}\boldsymbol{\beta}_1$$

and substituting $\boldsymbol{\beta}_2$ into the first equation we obtain

$$\boldsymbol{b}_1 - \boldsymbol{B}_{21}\boldsymbol{B}_{22}^{-1}\boldsymbol{b}_2 + (\boldsymbol{B}_{11} - \boldsymbol{B}_{21}\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{12})\boldsymbol{\beta}_1 = \mathbf{0}.$$

Now let us denote $\mathbf{b} = \boldsymbol{b}_1 - \boldsymbol{B}_{21}\boldsymbol{B}_{22}^{-1}\boldsymbol{b}_2$, $\mathbf{B} = \boldsymbol{B}_{11} - \boldsymbol{B}_{21}\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{12}$. As the matrix $\mathbf{B}$ is of full row rank, we obtain (see e.g. in [6], p. 124) the model

$$\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta}_1 = \mathbf{0}.$$

The matrix $\mathbf{B}$ we split into two submatrices $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2)$, where $\mathbf{B}_2$ is a $q - k_2 \times q - k_2$ matrix and moreover (without lose of generality) let it be a regular matrix. The vector $\boldsymbol{\beta}_1$ we split into two parts $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$ in such a way that $\boldsymbol{\gamma}_1$ is a $k_1 + k_2 - q$ dimensional vector and $\boldsymbol{\gamma}_2$ is a $q - k_2$ dimensional vector. The constraint on the parameters can be rewritten as

$$\mathbf{b} + \mathbf{B}_1\boldsymbol{\gamma}_1 + \mathbf{B}_2\boldsymbol{\gamma}_2 = \mathbf{0}.$$

From the last equation is

$$\boldsymbol{\gamma}_2 = -\mathbf{B}_2^{-1}\mathbf{b} - \mathbf{B}_2^{-1}\mathbf{B}_1\boldsymbol{\gamma}_1$$

and the model $\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \mathbf{b} + \mathbf{B}\boldsymbol{\beta}_1 = \mathbf{0}$ can be written as

$$\boldsymbol{Y} \sim \left(\begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{pmatrix}, \boldsymbol{\Sigma} \right), \qquad \mathbf{b} + \mathbf{B}_1\boldsymbol{\gamma}_1 + \mathbf{B}_2\boldsymbol{\gamma}_2 = \mathbf{0}.$$

Including the constraints into the model we obtain

$$\boldsymbol{Y} \sim \left(\begin{pmatrix} \gamma_1 \\ -\mathbf{B}_2^{-1}\mathbf{b} - \mathbf{B}_2^{-1}\mathbf{B}_1\gamma_1 \end{pmatrix}, \boldsymbol{\Sigma}\right),$$

or

$$\boldsymbol{Y} + \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_2^{-1}\boldsymbol{b} \end{pmatrix} \sim \left(\begin{pmatrix} \boldsymbol{I} \\ -\mathbf{B}_2^{-1}\mathbf{B}_1 \end{pmatrix}\gamma_1, \boldsymbol{\Sigma}\right).$$

Denoting $\boldsymbol{\beta}_{1,0} = \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_2^{-1}\boldsymbol{b} \end{pmatrix}$, $\boldsymbol{K}_1 = \begin{pmatrix} \boldsymbol{I} \\ -\mathbf{B}_2^{-1}\mathbf{B}_1 \end{pmatrix}$ and taken into account that $\boldsymbol{K}_1$ is a full rank matrix in columns, we obtain the model

$$\boldsymbol{Y} - \boldsymbol{\beta}_{1,0} \sim (\boldsymbol{K}_1\gamma_1, \boldsymbol{\Sigma}).$$

The former procedure is the proof of the next Lemma

Lemma 1 The regular regression model with type-II constraints

$$\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \quad \boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \mathbf{0}.$$

is equivalent to the (regular) model without constraints

$$\boldsymbol{Y} - \boldsymbol{\beta}_{1,0} \sim (\boldsymbol{K}_1\gamma_1, \boldsymbol{\Sigma}).$$

Appendix 3. MINQUE estimators

The MINQUE method of estimation is used to estimate the unknown components of the covariance matrix. Let us consider the model $\boldsymbol{Y} \sim (\boldsymbol{X\beta}, \boldsymbol{\Sigma})$ (model without constraints on the parameters). The model can be written equivalently as $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a random vector of errors. This vector is often of the form $\boldsymbol{\varepsilon} = \boldsymbol{U}_1\boldsymbol{\xi}_1 + ... + \boldsymbol{U}_p\boldsymbol{\xi}_p$, $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_p$ are known matrices, $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_p$ are random vectors of dimension $n_1, \ldots, n_p$ for which it holds

$$\mathcal{E}(\boldsymbol{\xi}_i) = \mathbf{0}, \quad \mathrm{cov}(\boldsymbol{\xi}_i) = \vartheta_i\boldsymbol{I}, \quad \text{where } \vartheta_i > 0 \text{ for } i = 1, \ldots, p$$

and

$$\mathrm{cov}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) = \mathbf{0} \quad \text{for } i, j = 1, 2, \ldots, p, \quad i \neq j.$$

This model is often called *model with mixed effects*. The covariance matrix of this model is

$$\mathrm{cov}(\boldsymbol{Y}) = \sum_{i=1}^{p} \vartheta_i\boldsymbol{U}_i\boldsymbol{U}_i' = \sum_{i=1}^{p} \vartheta_i\boldsymbol{V}_i.$$

Let us denote $\boldsymbol{S_W}$ the matrix which $(i,j)-$th element is $\{\boldsymbol{S_W}\}_{i,j} = \mathrm{Tr}\{\boldsymbol{WV}_i\boldsymbol{WV}_j\}$ and $\mathcal{M}(\boldsymbol{T})$ the linear space spanned by the columns of the matrix $\boldsymbol{T}$.

Theorem 6 Let us have a regular regression model without constraints $\boldsymbol{Y} \sim (\boldsymbol{X\beta}, \boldsymbol{\Sigma})$ and the covariance matrix $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i\boldsymbol{V}_i$. Let $\vartheta_1^0, \ldots, \vartheta_p^0$ are known (approximate) values of unknown components $\vartheta_1, \ldots, \vartheta_p$ and $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^0\boldsymbol{V}_i$, $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$, $\boldsymbol{g} = (g_1, \ldots, g_p)'$. Then

(i) The function $g(\boldsymbol{\vartheta}) = \boldsymbol{g}'\boldsymbol{\vartheta} = \sum_{i=1}^{p} g_i\vartheta_i$ is estimable using the MINQUE procedure iff

$$\boldsymbol{g} \in \mathcal{M}(\boldsymbol{S}_{(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+}),$$

where $\mathcal{P}_{\boldsymbol{X}} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ and $\boldsymbol{W}^+$ is the Moore-Penrose g-inversion of the matrix $\boldsymbol{W}$.

(ii) If the function $g(\cdot)$ satisfies the condition (i) then the MINQUE (Minimum norm quadratic unbiased estimator), see e.g. [8] of this function is

$$\widehat{\boldsymbol{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i\boldsymbol{Y}'(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{V}_i(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{Y},$$

where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ is the solution of the equation

$$\boldsymbol{S}_{(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+}\boldsymbol{\lambda} = \boldsymbol{g}.$$

If the matrix $\boldsymbol{S}_{(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+}$ is regular (i.e. positive definite) then the MINQUE estimator of the whole vector $\boldsymbol{\vartheta}$ exists and

$$\widehat{\boldsymbol{\vartheta}} = \boldsymbol{S}^{-1}_{(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+}\begin{pmatrix}\boldsymbol{Y}'(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{V}_1(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{Y}\\ \vdots \\ \boldsymbol{Y}'(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{V}_p(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+\boldsymbol{Y}\end{pmatrix}.$$

(iii) In the case of normally distributed random vector $\boldsymbol{Y}$, in addition, the covariance matrix of the estimator $\widehat{\boldsymbol{\vartheta}}$ is

$$\operatorname{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0) = 2\boldsymbol{S}^{-1}_{(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+}.$$

The proof can be see e.g. in [8], [6].

Lemma 2 Let $\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma})$, $\boldsymbol{b}+\boldsymbol{B}_1\boldsymbol{\beta}_1+\boldsymbol{B}_2\boldsymbol{\beta}_2 = \boldsymbol{0}$ be a regular regression model with type-II constraints, then

(i) $(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+ = \boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1$,

(ii) $(\mathcal{P}_{\boldsymbol{X}}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{X}})^+(\boldsymbol{Y} - \boldsymbol{\beta}_{1,0}) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \widehat{\boldsymbol{\beta}}_1)$,

where $\boldsymbol{K}_1, \boldsymbol{\beta}_{1,0}$ are from Lemma 1 and $\boldsymbol{Q}_{11}, \widehat{\boldsymbol{\beta}}_1$ are from Theorem 4 (assuming there $\boldsymbol{X} = \boldsymbol{I}$).
The proof can be found in [6], p.194.

Note Relations (i) and (ii) hold also if we instead of the matrix $\boldsymbol{\Sigma}$ consider the matrix $\boldsymbol{\Sigma}_0$, but we need this replacement realize also in expressions for the matrix $\boldsymbol{Q}_{11}$ and for the estimator $\widehat{\boldsymbol{\beta}}_1$.

Lemma 3 Let $\overline{\boldsymbol{Y}} \sim (\boldsymbol{\beta}_1, \frac{1}{m}\boldsymbol{\Sigma})$, $\boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \boldsymbol{0}$ be a regular regression model with type-II constraints, then

(i) $(\mathcal{P}_{\boldsymbol{K}_1}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{K}_1})^+ = \boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1$,

(ii) $(\mathcal{P}_{\boldsymbol{K}_1}\boldsymbol{\Sigma}_0\mathcal{P}_{\boldsymbol{K}_1})^+(\overline{\boldsymbol{Y}} - \boldsymbol{\beta}_{1,0}) = \boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{Y}} - \widehat{\boldsymbol{\beta}}_1)$,

where $\boldsymbol{K}_1$, $\boldsymbol{\beta}_{1,0}$ are from Lemma 1, $\widetilde{\boldsymbol{Q}}_{11}$ is from Theorem 5 (assuming there $\boldsymbol{X} = \boldsymbol{I}$) and $\widehat{\boldsymbol{\beta}}_1$ is also from Theorem 5.
The proof follows from Lemma 1.

Note Relations (i) and (ii) again hold also if we instead of the matrix $\boldsymbol{\Sigma}$ consider the matrix $\boldsymbol{\Sigma}_0$, but we need this replacement realize also in expressions for the matrix $\boldsymbol{Q}_{11}$ and for the estimator $\widehat{\boldsymbol{\beta}}_1$.
Using the connection of Lemma 2, Theorem 4 and utilizing Lemma 1 we obtain the next theorem.

Theorem 7 Let $\boldsymbol{Y} \sim (\boldsymbol{\beta}_1, \boldsymbol{\Sigma})$, $\boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \boldsymbol{0}$ be a regular regression model with type-II constraints and let the covariance matrix $\boldsymbol{\Sigma} = \sum_{i=1}^p \vartheta_i\boldsymbol{V}_i$. Let us denote $\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_i^0\boldsymbol{V}_i$, $\boldsymbol{Q}_{11}, \widehat{\boldsymbol{\beta}}_1$ are from Theorem 3 (assuming there $\boldsymbol{X} = \boldsymbol{I}$). Then

(i) The function $g(\boldsymbol{\vartheta}) = \boldsymbol{g}'\boldsymbol{\vartheta} = \sum_{i=1}^p g_i\vartheta_i$ is estimable using the MINQUE procedure iff

$$\boldsymbol{g} \in \mathcal{M}(\boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}).$$

(ii) If the function $g(\cdot)$ satisfies the condition (i) then the MINQUE of this function is

$$\widehat{\boldsymbol{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^p \lambda_i(\boldsymbol{Y} - \widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \widehat{\boldsymbol{\beta}}_1),$$

where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ is the solution of the equation

$$\boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}\boldsymbol{\lambda} = \boldsymbol{g}.$$

If the matrix $\boldsymbol{S}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}$ is regular (i.e. positive definite) then the MINQUE estimator of the whole vector $\boldsymbol{\vartheta}$ exists and

$$\widehat{\boldsymbol{\vartheta}} = \boldsymbol{S}^{-1}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)} \begin{pmatrix} (\boldsymbol{Y}-\widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y}-\widehat{\boldsymbol{\beta}}_1) \\ \vdots \\ (\boldsymbol{Y}-\widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_p\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y}-\widehat{\boldsymbol{\beta}}_1) \end{pmatrix}.$$

(iii) In the case of normally distributed random vector $\boldsymbol{Y}$, in addition, the covariance matrix of the estimator $\widehat{\boldsymbol{\vartheta}}$ is

$$\operatorname{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0) = 2\boldsymbol{S}^{-1}_{(\boldsymbol{B}_1'\boldsymbol{Q}_{11}\boldsymbol{B}_1)}.$$

Finally we reformulate Theorem 7 for the case of replicated model

Theorem 8 Let $\underline{\boldsymbol{Y}} \sim [(\boldsymbol{1}_m \otimes \boldsymbol{I}), \boldsymbol{I}_m \otimes \boldsymbol{\Sigma}]$ with constraints $\boldsymbol{b} + \boldsymbol{B}_1\boldsymbol{\beta}_1 + \boldsymbol{B}_2\boldsymbol{\beta}_2 = \boldsymbol{0}$ be a replicated regular regression model with type-II constraints and the matrix $\boldsymbol{\Sigma} = \sum_{i=1}^p \vartheta_i \boldsymbol{V}_i$. Let be $\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_i^0 \boldsymbol{V}_i$, $\overline{\boldsymbol{Y}} = \frac{1}{m}\sum_{j=1}^m \boldsymbol{Y}_j$, $\boldsymbol{S} = \frac{1}{m-1}\sum_{j=1}^m (\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})(\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})'$, $\widetilde{\boldsymbol{Q}}_{11}$, $\widehat{\boldsymbol{\beta}}_1$ are from Theorem 4. Then

(i) The function $g(\boldsymbol{\vartheta}) = \boldsymbol{g}'\boldsymbol{\vartheta} = \sum_{i=1}^p g_i\vartheta_i$ is estimable using the MINQUE procedure iff

$$\boldsymbol{g} \in \mathcal{M}\left((m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1)}\right).$$

(ii) If the function $g(\cdot)$ satisfies the condition (i) then the MINQUE of this function is

$$\widehat{\boldsymbol{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^p \lambda_i (m-1)\operatorname{Tr}\{\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}_0^{-1}\boldsymbol{S}\} + m(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1),$$

where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ is the solution of the equation

$$\left[(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1)}\right]\boldsymbol{\lambda} = \boldsymbol{g}.$$

If the matrix $(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1)}$ is regular (i.e. positive definite) then the MINQUE estimator of the whole vector $\boldsymbol{\vartheta}$ exists and

$$\widehat{\boldsymbol{\vartheta}} = \left[(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1)}\right]^{-1}\widehat{\gamma},$$

with

$$\widehat{\gamma} = \begin{pmatrix} (m-1)\operatorname{Tr}\{\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}\boldsymbol{S}\} + m(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1) \\ \vdots \\ (m-1)\operatorname{Tr}\{\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}\boldsymbol{S}\} + m(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1)'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{Y}}-\widehat{\boldsymbol{\beta}}_1). \end{pmatrix}$$

(iii) In the case of normally distributed random vector $\boldsymbol{Y}$, in addition, the covariance matrix of the estimator $\widehat{\boldsymbol{\vartheta}}$ is

$$\operatorname{cov}(\widehat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0) = 2\left[(m-1)\boldsymbol{S}_{\boldsymbol{\Sigma}_0^{-1}} + \boldsymbol{S}_{(\boldsymbol{B}_1'\widetilde{\boldsymbol{Q}}_{11}\boldsymbol{B}_1)}\right]^{-1}.$$

The detailed proof is in [7]. The outline of the proof follows the next steps. First based on Lemma 1 using $\boldsymbol{K}_1, \boldsymbol{\beta}_{1,0}$ we transform the model with constraints on the parameters to an equivalent model without constraints on the parameters

$$\underline{\boldsymbol{Y}} - (\boldsymbol{1} \otimes \boldsymbol{I})\boldsymbol{\beta}_{1,0} \sim \left[(\boldsymbol{1} \otimes \boldsymbol{K}_1)\gamma, \sum_{i=1}^p \vartheta_i(\boldsymbol{I} \otimes \boldsymbol{V}_i)\right].$$

In that replicated model without constraints on the parameters we estimate the unknown parameters $\widehat{\boldsymbol{\vartheta}}$ and using Lemma 3 finally we obtain the desired MINQUE estimators.

Appendix 4. Kenward and Roger's procedure

The procedure published in [4] by Kenward and Roger serves to construction of (approximate) confidence regions for parameters (and for linear combinations of parameters) in regression model $\boldsymbol{Y} \sim (\boldsymbol{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, where $\boldsymbol{X}$ is a $(n \times k)$ full column rank matrix, $\boldsymbol{Y}$ has multivariate normal distribution and the covariance matrix $\boldsymbol{\Sigma}$ is assumed to be a function of an (unknown) vector of parameters $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_r)'$. Here let us consider that $\boldsymbol{\Sigma} = \sum_{i=1}^{r} \vartheta_i \boldsymbol{V}_i$ ($\boldsymbol{V}_i$ are known matrices). Further let us denote $\boldsymbol{\Phi}(\boldsymbol{\vartheta}) = \{\boldsymbol{X}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\boldsymbol{X}\}^{-1}$ and $\widehat{\boldsymbol{\Phi}} = \boldsymbol{\Phi}(\widehat{\boldsymbol{\vartheta}})$. The covariance matrix of $\widehat{\boldsymbol{\vartheta}}$ let be $\boldsymbol{W}$. An adjusted estimator of the small sample covariance matrix of $\boldsymbol{\beta}$ is suggested in [4]. Its form is

$$\widehat{\boldsymbol{\Phi}}_A = \widehat{\boldsymbol{\Phi}} + 2\widehat{\boldsymbol{\Phi}} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} \{\boldsymbol{W}\}_{i,j} (\boldsymbol{Q}_{i,j} - \boldsymbol{P}_i \widehat{\boldsymbol{\Phi}} \boldsymbol{P}_j) \right\} \widehat{\boldsymbol{\Phi}},$$

where

$$\boldsymbol{P}_i = \boldsymbol{X} \frac{\partial \boldsymbol{\Sigma}^{-1}}{\partial \vartheta_i} \boldsymbol{X}, \quad \boldsymbol{Q}_{i,j} = \boldsymbol{X} \frac{\partial \boldsymbol{\Sigma}^{-1}}{\partial \vartheta_i} \boldsymbol{\Sigma} \frac{\partial \boldsymbol{\Sigma}^{-1}}{\partial \vartheta_j} \boldsymbol{X}$$

Let us consider l linear combinations of vector $\boldsymbol{\beta}$ elements: $\boldsymbol{L}'\boldsymbol{\beta}$, where $\boldsymbol{L}$ is a known $(k \times l)$ matrix. Denoting

$$F = \frac{1}{l} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \boldsymbol{L} (\boldsymbol{L}' \widehat{\boldsymbol{\Phi}}_A \boldsymbol{L})^{-1} \boldsymbol{L}' (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$$

in [4] is shown that λF has approximate Fisher-Snedecor F distribution with l and m^* degrees of freedom, where

$$\lambda = \frac{m^*}{E^*(m-2)}, \quad m^* = 4 + \frac{l+2}{l\varrho - 1}, \quad \varrho = \frac{V^*}{2(E^*)^2},$$

$$V^* = \frac{2}{l} \, \frac{1 + c_1 B}{(1 - c_2 B)^2 (1 - c_3 B)}, \quad E^* = \left(1 - \frac{A_2}{l}\right)^{-1},$$

$$c_3 = \frac{l + 2 - g}{3l + 2(1-g)}, \quad c_2 = \frac{l - g}{3l + 2(1-g)}, \quad c_1 = \frac{g}{3l + 2(1-g)},$$

$$g = \frac{(l+1)A_1 - (l+4)A_2}{(l+2)A_2}, \quad B = \frac{1}{2l}(A_1 + 6A_2),$$

$$A_1 = \sum_{i=1}^{r} \sum_{j=1}^{r} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta\Phi P}_i \boldsymbol{\Phi}\} \operatorname{Tr}\{\boldsymbol{\Theta\Phi P}_j \boldsymbol{\Phi}\}, \quad A_2 = \sum_{i=1}^{r} \{\boldsymbol{W}\}_{i,j} \operatorname{Tr}\{\boldsymbol{\Theta\Phi P}_i \boldsymbol{\Phi\Theta\Phi P}_i \boldsymbol{\Phi}\},$$

$$\boldsymbol{\Theta} = \boldsymbol{L}(\boldsymbol{L}'\boldsymbol{\Phi}\boldsymbol{L})^{-1}\boldsymbol{L}'.$$

In the last three relations we substitute the parameters $\vartheta_1, \ldots, \vartheta_r$ by their estimators $\widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_r$. The suggested (approximate) $(1-\alpha)$ confidence region for l linear combinations of $\boldsymbol{\beta}$: $\boldsymbol{L}'\boldsymbol{\beta}$ is

$$\left\{ \boldsymbol{\beta} : \quad F \leq \frac{1}{\lambda} F_{l,m^*}(1-\alpha) \right\},$$

where $F_{l,m^*}(1-\alpha)$ is the $(1-\alpha)$ quantile of the Fisher-Snedecor F distribution with l and m^* degrees of freedom.

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II

Principles of Frequent Subgraph Mining in Networks: Methods, Related Topics and Applications

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1 Introduction

In present days, human society is faced with a permanent production of large amount of data, the storage, processing, evaluation and interpretation of which costs a considerable effort. A portion of these data is represented or stored in the form of two-entity links, such as friend-relation in social associations. A common abstraction for such data types represents a network topology or graph. Networks or graphs (both notions are synonyms) are combinatorial structures comprising of nodes (i.e. entities, objects) connected by links. Instances of networks are frequently present in many application domains, e.g. transportation, electrical engineering, social science, chemistry, biology, scientometry, etc. Typical research directions regarding the real-world networks are aimed at network structure and dynamics. Since the real-world networks are large (from thousands to more than millions of nodes and links), the corresponding analysis should use rather smart methods than brute-force. Such an approach is closely related to the knowledge discovery and data mining methods and that is why it is called the *graph mining*. More precisely, the term graph mining refers a collection of algorithms, techniques and tools that deals with analysis of real-world graphs and their properties, with prediction how the topology (or the structure) of a given graph might affect applications and with development of realistic models that match important (or specific) features of real-world graphs (Chakrabarti, 2011). Similarly as in case of data mining, usually used methods in the graph mining come from the machine learning, statistic, the graph theory and, eventually, the computational intelligence. The current chapter is focused on one of the core graph mining applications, the *frequent subgraph mining* (shortly FSM). Such a topic is aimed at the problem of searching for subgraphs with a significant frequency of occurrence in a given set of graphs (Chakrabarti, 2011). The original motivation of such a task comes from chemistry, e.g. (Aggarwal, 2015), pp. 575-576. An example is shown in Fig. 1, where the pyridine fragment (i.e. the pyridine without all hydrogen atoms) is occurred four-times in set of 3 compounds containing 4,4'-bipyridine, picolinic acid and nicotinic acid, respectively. It means that the pyridine fragment is the "largest" and the most frequent substructure being occurred in all three compounds. Similar questions are typically very useful in chemical datasets consisting of many different molecules (Chakrabarti, 2011).

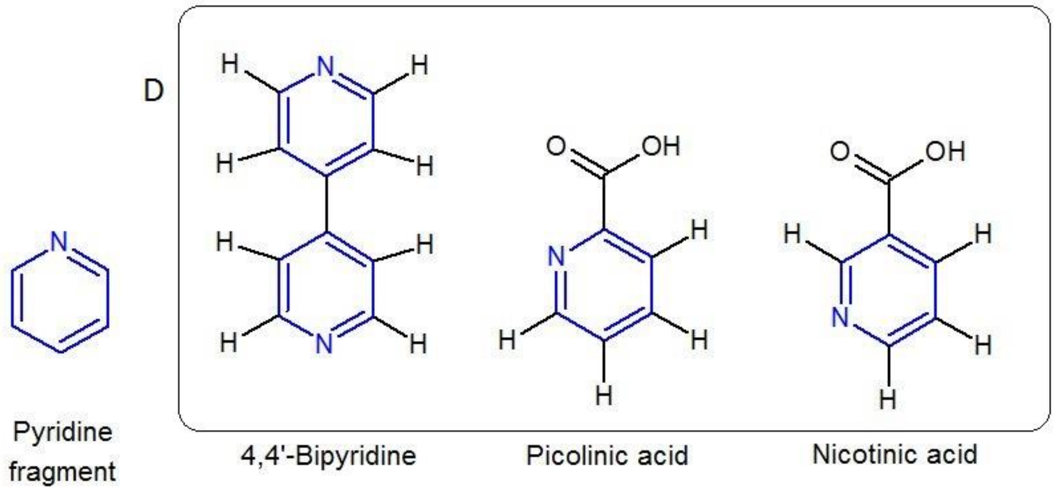

Fig. 1. *Let D be a dataset of 3 chemical compounds, namely 4,4'-bipyridine, picolinic acid and nicotinic acid. The fragment of pyridine (the left-most compound, i.e. the pyridine without hydrogens) is the substructure (or "pattern") which occurs in all compounds from D. Note that it is totally occurred four-times, since it is present twice in 4,4'-bipyridine. The corresponding patterns are colored by blue.*

Such a substructural property, however, is not limited to chemistry. Similar problems are also addressed in biology (specifically bioinformatics), social sciences, security, etc.; we refer to (Ramraj & Prabhakar, 2015) for details.

The frequent subgraph mining (abbreviated to FSM) problem can be viewed from many perspectives. The first, the problem itself is algorithmically difficult because of the the NP-completeness of the graph isomorphism (see details below). It means, in turn, that design of effective algorithms is not easy. Despite this difficulty, many algorithmic approaches are currently known and these methods are usually able to solve the FSM in a satisfactory manner. Another perspective on the FSM may regard the FSM software tools and their experimental performance comparison. The third aspect resides in the fact that the FSM algorithms are fundamental building blocks for graph mining algorithms (Aggarwal, 2015), p. 575. The reason is that the property "being subgraph of a graph" is subsumed in many other graph mining scenarios. Such way, the FSM problem is closely related to other graph mining problems, such as graph kernels, graph classification, graph clustering, etc. The last point of view we mention here is the fact that the FSM algorithms are applicable in many specific domains (e.g. chemistry, sociology, biology). Some of these domains, however, require appropriate adjusting of the original FSM problem. It leads to many diverse variations of the aforementioned problem.

Since it is not realistic to survey all aspects entirely, the present chapter is primarily aimed at such variations of the FSM problem which are related to the current research taking place at Faculty of Chemical and Food Technology, Slovak University of Technology in Bratislava. Therefore, it follows two principal directions:

- the coordination chemistry with organic/inorganic ligands and
- the structure of real-world networks (including the protein-protein interaction networks).

In order to maintain the research efficiency, two cutting-edge software tools are used: *CCDC ConQuest* and *Orca*, respectively. In this chapter, their common principle is described and their mutual differences are explained, as well. Main aspects of the research taking place are also specified.

The chapter is organized as follows. Definitions and preliminaries of the FSM problem are mentioned in Sect. 2. The algorithmic principles are described in Sect. 3. The subgraph mining problem in chemical datasets is introduced in Sect. 4. The graphlet-based comparison of real-world networks is described in Sect. 5.

2 Definitions and the problem statement

A graph is an abstract framework for networks with nodes and their connections. The definitions of graphs, subgraphs and related terms are introduced in this section. A general case of terms and concepts (including labelled and directed graphs/subgraphs) is considered in this section since their specific instances, sufficient in some applications, are formulated in straightforward manner. Alternative restrictions of the general case are commented further in this chapter.

Graph. A *labelled graph* is a 6-tuple $G=(V, E, \mathcal{L}_V, \mathcal{L}_E, \sigma, \tau)$, where V is a nonempty set of vertices, $E \subseteq V \times V$ is a set of edges, $\mathcal{L}_V$ is a set of vertex labels, $\mathcal{L}_E$ is a set of edge labels, σ:

$V \rightarrow \mathcal{L}_V$ is a *vertex label function* which maps vertices to their labels and $\tau: E \rightarrow \mathcal{L}_E$ is an edge label function which is also a mapping from E to $\mathcal{L}_E$.

Labelled graphs are referred to as *attributed graphs* (Dhiman & Jain, 2016), as well. In Fig. 1., vertex attributes are either atoms H, C (not visible), N, O or groups OH. The edge attributes may be either the length of bonds or their multiplicity.

A labelled graph is said to be *directed* if for all e∈E, e is an ordered pair of vertices, denoted by e=(u, v). A labelled graph is said to be *undirected* if for all e∈E, e is an unordered pair of vertices, denoted by e={u, v}. A *path* in G, denoted by P, is a sequence of vertices which can be ordered such that two vertices correspond to an edge of E iff (i.e. if and only if) they are consecutive in the list. If u, v are vertices such that u≠v, a path from u to v is denoted by P(u, v), i.e. u and v are two endpoints of P(u,v). A graph is said to be *connected* if for all pairs of vertices u, v ∈ V, u≠v, there is either a path P(u, v) or P(v, u) or both. G is *disconnected* otherwise. A path P(u, v) with at least 3 vertices is said to be a *cycle* if u=v.

Subgraph. Let $G_1=(V_1, E_1, \mathcal{L}_{V1}, \mathcal{L}_{E1}, \sigma_1, \tau_1)$ and $G_2=(V_2, E_2, \mathcal{L}_{V2}, \mathcal{L}_{E2}, \sigma_2, \tau_2)$ be two graphs. G_1 is a *subgraph* of G_2, if G_1 satisfies:

i. $V_1 \subseteq V_2$ and for all $u \in V_1$ it holds $\sigma_1(u)=\sigma_2(u)$;
ii. $E_1 \subseteq E_2$ and for all $(u, v) \in E_1$ it holds $\tau_1(u, v)=\tau_2(u, v)$.

Induced subgraph. Given graphs G_1 and G_2, we say that G_1 is an *induced subgraph* of G_2 if the following conditions are satisfied:

i. G_1 is a subgraph of G_2;
ii. for all $u, v \in V_1$, $(u, v) \in E_1$ iff $(u, v) \in E_2$.

A *cyclic graph* is a graph that at least one its subgraph is a cycle; equivalently, a graph containing at least one cycle. A graph is said to be *acyclic* otherwise. A connected acyclic graph is said to be a *tree*.

A topological "equivalence" of two graphs is defined through mapping referred to as a graph isomorphism. The corresponding formalism is as follows.

Graph Isomorphism. A graph $G_1=(V_1, E_1, \mathcal{L}_{V1}, \mathcal{L}_{E1}, \sigma_1, \tau_1)$ *is isomorphic to a graph* $G_2=(V_2, E_2, \mathcal{L}_{V2}, \mathcal{L}_{E2}, \sigma_2, \tau_2)$ iff there exists a bijection $f: V_1 \rightarrow V_2$ such that:

i. for all $u \in V_1$, $\sigma_1(u)=\sigma_2(f(u))$;
ii. for all $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$;
iii. for all $(u, v) \in E_1$, $\tau_1(u, v)=\tau_2(f(u), f(v))$.

If G_1 and G_2 are isomorphic then we can write $G_1 \cong G_2$ and the bijection f is an *isomorphism* between G_1 and G_2.

Note that the isomorphic relation represents the equivalence on graphs. One of the crucial concepts which the FSM is based on is the subgraph isomorphism.

Subgraph Isomorphism. A graph G_1 is *subgraph isomorphic to a graph* G_2 iff there exists a subgraph H of G_2 such that $G_1 \cong H$.

A subgraph isomorphic to a given graph is also called an *occurrence* (Dhiman & Jain, 2016). In order to explain the subgraph isomorphism, recall Fig. 1. In it, one can understood molecules as labelled graphs with vertices labelled by elements C (Carbons are not visible in the figure), H, N, O or group OH and chemical bonds (single or double bonds) are represented by edges. Note that in terms of graph theory, the pyridine fragment is the subgraph isomorphic to itself in all three compounds of the dataset D.

There are two principal formulations of the FSM problem. The difference between them is in the fact of whether the input (except a "small" graph being searched) is the set of many graphs or a single graph only. The former case is introduced as follows.

FSM in a graph database. Let $d>1$ be an integer and let $|X|$ denote the cardinality of a finite set X. Let $\Gamma=\{G_1, G_2, \ldots, G_d\}$ be a collection of graphs (a graph database) and let t, $0<t\leq 1$, be a real number called a *support threshold*. If g is a graph then $\Delta(g, \Gamma)$ denotes a set of graphs $G_i \in \Gamma$ where g occurs at least once (the number of occurrence g in a single G_i is not relevant), formally: $\Delta(g, \Gamma) = \{ G_i \mid g \subseteq G_i \}$. Thus, the *support* of g is defined by

$$Supp(g, \Gamma) = \frac{|\Delta(g, \Gamma)|}{d}. \qquad (1)$$

and g is a *frequent subgraph* iff $Supp(g, \Gamma) \geq t$.

The FSM in a single graph represents a more complex problem since a multiple occurrence of a given (small) graph should be counted. Different occurrences of a given subgraph may overlap, leading to possible ambiguous counting. Establishing an appropriate support measure in the single-graph setting is thus quite difficult issue. Various strategies are discussed e.g. in (Flores-Garrido, 2015), pp. 13-16. A simple but cardinal idea is due to B. Bringmann and S. Nijssen (Bringmann & Nijssen, 2008). It is also adopted in this chapter. The aforementioned idea is based on the minimum number of unique isomorphism mappings over all the subgraph vertices where a given subgraph is isomorphically equivalent to an input graph. The corresponding definition is as follows.

FSM in a single graph. Let k≥1 be an integer and let G be a graph. If g is a subgraph isomorphic to G, then let f denote an isomorphism which maps g to one its occurrence in G, i.e. f(g) is an image of g. Clearly, if u ∈ V(g), then f(u) ∈ f(V(g)). The *support* of g in G is defined by

$$Supp_1(g, \mathrm{G}) = \min_{u \in V(g)} |\{f(u) : \ f(g) \subseteq G\}| \, . \qquad (2)$$

Graph g is a *frequent subgraph in* (a single graph) G iff $\mathrm{Supp}_1(\mathrm{g, G}) \geq \mathrm{k}$.
Observe that, according to the above definition, the support of g in G is the minimum number of such "vertex-isomorphic images" that there exists a subgraph isomorphic to g in G, where the minimum is taken over all vertices of g.
Example of a support determination for a single graph is illustrated in Fig. 2.

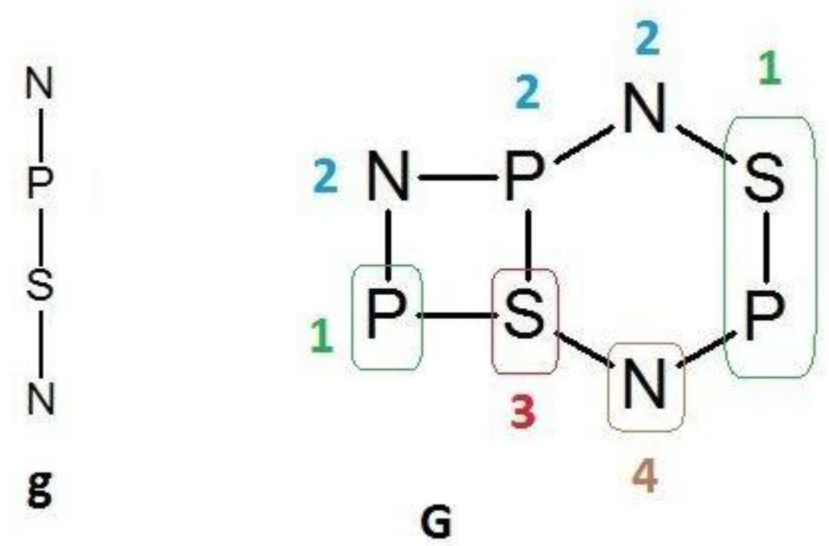

Fig. 2. The graph g is a path N-P-S-N (left) and the graph G (consisting of 2 cycles) is shown on the right. In both graphs, only vertices are attributed; the edges are without attributes. There are overall 4 different isomorphisms of g into subgraphs in G. The number associated to each vertex of G represents how many isomorphisms do map any vertex of g to that vertex in G. The minimum number of images of any vertex in g is attained for S. Clearly, S has two images in G and hence $Supp_1(g, G)=2$. (Note that the total number of isomrphisms is not taking into account.)

Modifications of the FSM problem, namely the subgraph mining and the graph frequency counting, are described in sections 4 and 5, respectively.

3 The algorithmic perspective

Although the FSM in a graph database represents the simpler problem than that for a single graph, it is still far from easy one. Due to the fact that the subgraph isomorphism problem is known to be NP-complete (Jiang, 2011), the FSM problem in a graph database is algorithmically difficult. It means that each naïve algorithm requires exponential time complexity with respect to the number of vertices of the input graph. Fortunately, there are algorithms that perform better than the naïve ones, they are surveyed e.g. in, (Flores-Garrido, 2015), (Jiang, 2011), or (Ramraj & Prabhakar, 2015). Such usable algorithms are primarily based on the exhaustive searching (Dhiman & Jain, 2016). The common principle of the FSM algorithms is based on the following scheme (Mrzic, et al., 2018).

1. Candidate generation
2. Pruning
3. Support computation

Such a sequence is repeated until no more frequent subgraph exists.

The step regarding the candidate generation is discussed e.g. in (Ramraj & Prabhakar, 2015) including alternatives such as “Apriori-based” or “Pattern-growth”.

If one considers the FSM in a single graph, then recall that the principal difference is laid in the possible overlapping of subgraphs. As shown in Fig. 2, searching the path N-P-S-N (denoted by g) in the graph G may leads to an ambiguity. Note e.g. that both vertices P and S occur in g once however, P occurs three times in G while S occurs twice! The correct solution of the problem is provided by the definition of the *anti-monotonicity property*, also referred to as the *Downward Closure Property* (shortly DCP). It is as follows.

Anti-monotonicity property. Given a graph G and a property Q. Q is *anti-monotone* iff for all subgraphs $g \subseteq G$ such that g satisfies Q implies that all subgraphs of g satisfy Q.

Clearly, Q refers to the subgraph being frequent for our purpose. Moreover, it can be shown that the definition of support given by eq. (2) satisfies the anti-monotonicity property. Maintaining the anti-monotonicity during the computation of a FSM algorithm is of great importance because it helps in pruning and, without it, the exhaustive search is unavoidable (Dhiman & Jain, 2016).

4 The subgraph mining and the ConQuest software

A simpler modification of the FSM is the *subgraph mining problem* (shortly SMP). In SMP, neither frequency of subgraphs nor the support threshold is considered. Despite such a reduction, the SMP in a graph database play an important role in chemistry. The motivation comes from the question of whether a given chemical substructure occurs in a dataset containing a huge number of chemical compounds.

The ConQuest is the commercial software developed and provided by the Cambridge Crystallographic Data Centre (CCDC, 2020) which is widely exploited at the Department of Inorganic Chemistry, Institute of Inorganic Chemistry, Technology and Materials, Faculty of Chemical and Food Technology, SUT in Bratislava. The ConQuest represents the searching application for the Cambridge Structural Database (CSD). The database is repeatedly updated and currently contains more than one million crystal structures. The ConQuest together with CSD represents one of the top-level software packages in the field. According to the taxonomy above, it solves the subgraph mining problem in a graph database although the subgraph searched is not to be necessarily the most frequent. Queries in ConQuest are input via the graphical interface; see Fig 3 and Fig. 4, respectively. Various extensions of a basic single-subgraph query are available, e.g. a query searching for more subgraphs or a query searching for subgraphs in compounds without “forbidden” patterns. Thank to very flexible search algorithm, including also the logical operations (and/or/not) with queries, the great number of the found results can be often reduced to acceptable number by a few-step adjusting of the input queries.

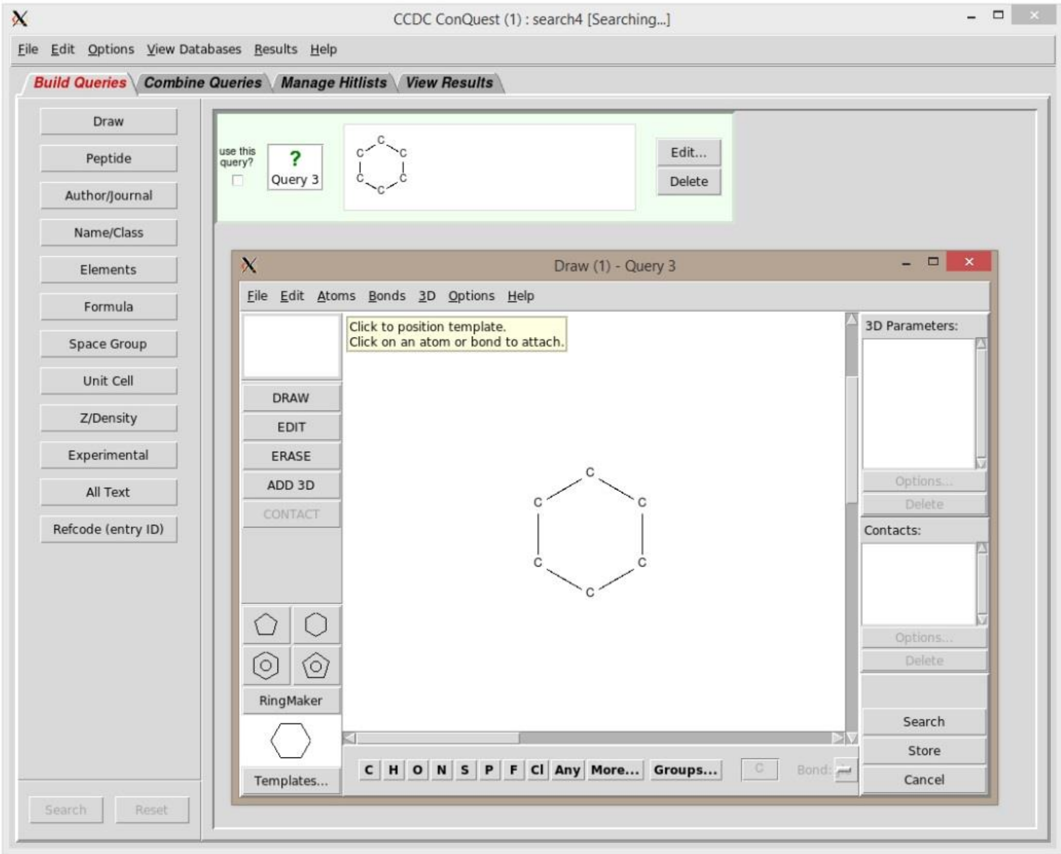

Fig. 3. Input of the query on the six-carbon single-bond ring (a fragment of cyclohexane without hydrogen atoms) in ConQuest.

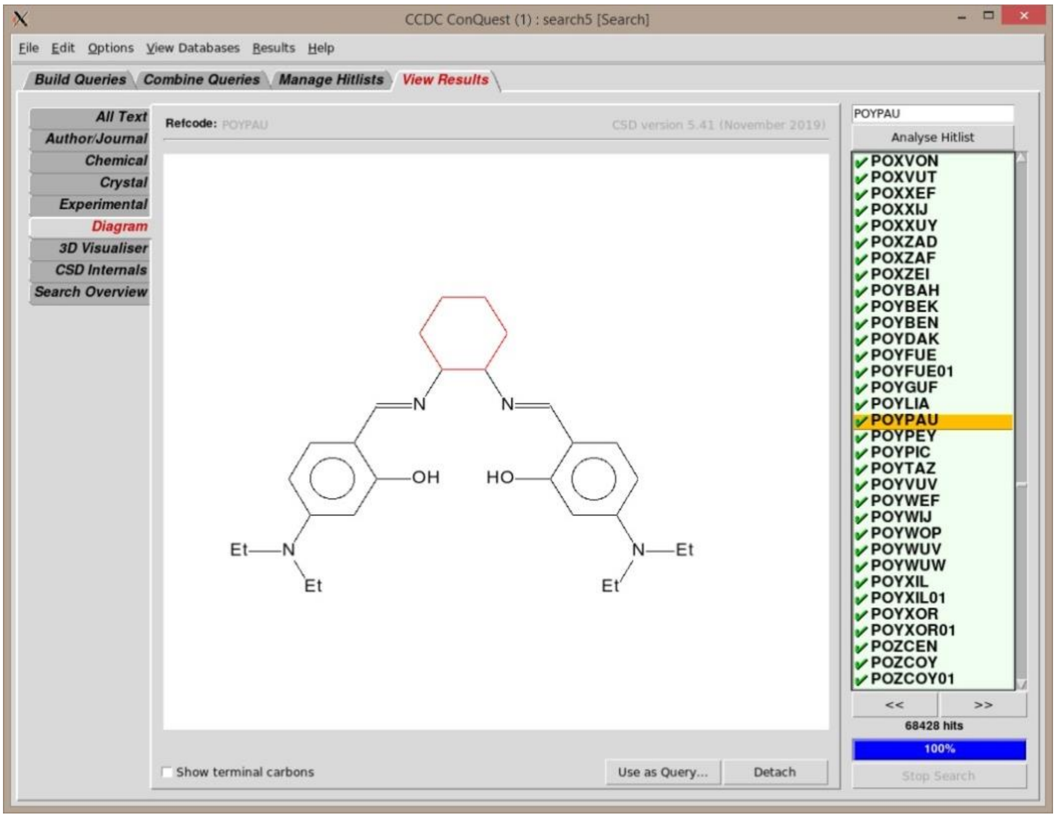

Fig. 4. The result of the query from Fig. 3. The subgraph containing the six-carbon single-bond ring subgraph is marked by the red color.

The ConQuest is very helpful for chemists for checking the already-prepared and published chemical structures containing selected structural motifs. Especially, coordination chemistry research needs to know all available combinations of more organic/inorganic ligands with various central atoms, to plan alternative syntheses of new complexes. From this point of view, a great advantage of CSD are the stored references of originally published papers.

5 Graphlets and the Orca software

Another variation of the FSM problem represents the searching for graphlets in a single graph.

Graphlet. Given a graph G, a *graphlet* is a connected induced subgraph of G with at most 5 vertices.

Two graphlets are the same if there exists an isomorphism which maps one graphlet to another one. Two different occurrences of the same graphlet are usually referred to as its copies. There is one graphlet with 2 vertices, there are 2 graphlets with 3 vetices, 6 graphlets with 4 vertices and 21 graphlets with 5 vertices, respectively. All graphlets with 2, 3 and 4 vertices are shown in Fig. 5. The graphlet mining problem is frequently studied in graphs without edge attributes, while vertices are usually attributed.

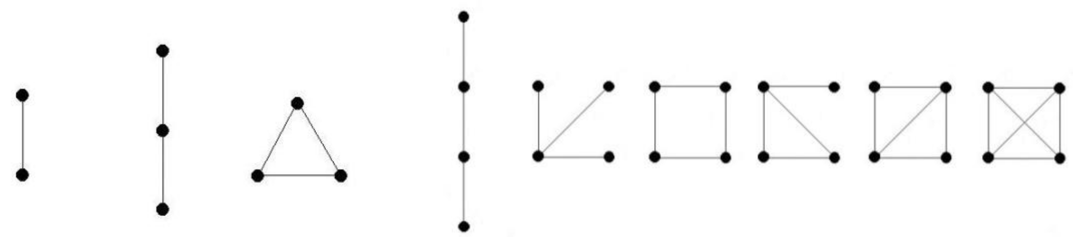

Fig. 5. All graphlets with 2, 3 and 4 vertices (from the left to the right). Both vertices and edges are without labels. These graphlets are denoted by g_1, g_2, ..., g_9 (from the left to the right).

Graphlets were introduced by N. Pržulj (Pržulj, Corneil, & Jurisica, 2004) for the purpose of the biological network comparison. More precisely, interactions among proteins in organisms are usually modelled by graphs, referred to as *protein-protein interaction networks* (shortly PPI networks). Studying similarities among PPI networks or, on the contrary, searching for anomaly among them is a challenging research problem with application in e.g. biomedicine. The graphlet-based method of the network comparison relies on frequency analysis of graphlets occurring in a given network. It turns out that the structure comparison of two networks is reduced to the comparison of two statistical distributions; see (Pržulj, 2007) for details. The most important part of the described process lays in the searching for graphlets and their frequency counting.

A few of software programs are currently available for the graphlet searching and counting, e.g. FANMOD, GraphCrunch, RAGE, Orca (Hočevar & Demšar, 2014). According to their performance tests published in (Hočevar & Demšar, 2014), it seems that Orca outperforms all other programs. The reason for this is in the sophisticated optimization of searching based on the system of linear equations for "vertex-orbit" (or "edge-orbit") counts (Hočevar & Demšar, 2016).

The Orca is a freeware which is currently used by the authors of this chapter to conduct the research aimed at comparison of real-world or artificially generated networks. Yet published papers regarding such a research topic are (Nehéz & Lelovský, 2018) and (Nehéz M. , 2018), respectively. Due to the fact that Orca is a console application and it provides neither graphic user interface nor choices for input setting, the authors are also developing software tools that would enrich the Orca functionality to basic statistical analysis of graphlet distributions or generation of artificial graphs. It is proposed that the future research directions would be aimed at improving related theoretical methods and, from the experimental perspective, the graphlet-based analysis of biological or social networks.

Essential parts of the process regarding the graphlet-based analysis of networks are illustrated in following examples.

Example 1. Let $n \geq 1$ be an integer and let $r>0$ be fixed real number referred to as *radius*. A 2-dimensional *random geometric graph* RGG(n, r) consits of a sequence $(x_1, x_2, ..., x_n)$ of independent and uniformly distributed points in the plane $[0, 1]\times[0, 1]$ such that all pairs of points x_i, x_j are adjacent if their mutual Euclidean distance is less or equal to r. Properties of random geometric graphs are examined e.g. in (Dall & Christensen, 2002). As resulted in (Pržulj, Corneil, & Jurisica, 2004), there is a resemblance between the random geometric graphs and protein-protein interaction networks modeling interactome of organisms. In order to generate random geometric graphs artificially, the authors of this chapter exploited the NetworkX tool accessed in (NetworkX, 2019). Generated networks are visualised by a software tool written in Python (Anaconda Distribution, 2019). Examples of two such graphs are illustrated in Fig. 6.

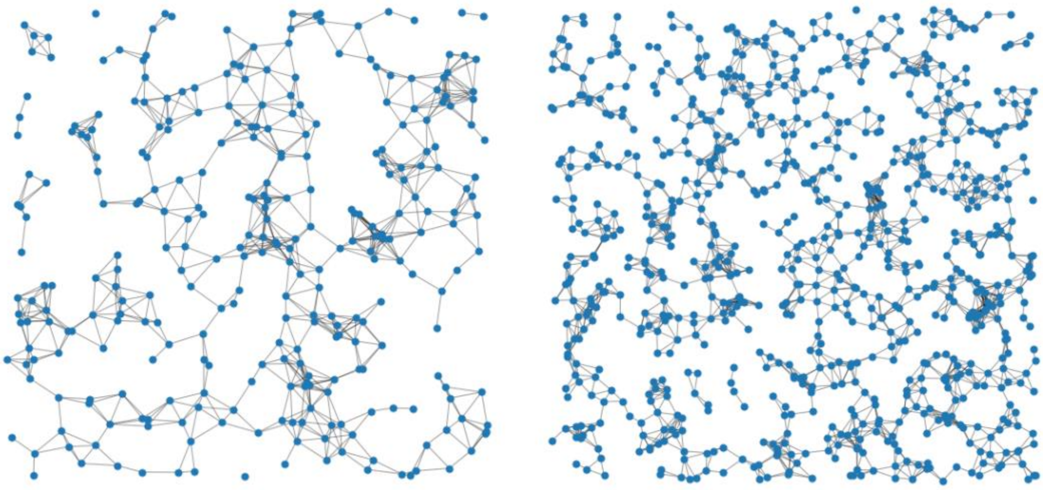

Fig. 6. Two examples of 2-dimensional random geometric graphs. The first graph (left) has 300 vertices, expected number of edges 900 and the radius r=0.07979. The second graph (right) has 800 vertices, expected number of edges 2400 and the radius r=0.04886.

The graphlet frequency counting process is illustrated by the following example.

Example 2. By the usage of the software tool mentioned in the previous example, we generated a collection of 20 random geometric graphs with 8000 vertices and with the radius r=0.01557. The expected number of edges in these graphs was 24000. In order to count the graphlet frequency, the Orca tool was exploited. These frequencies were very similar in all aforementioned graphs. The resulted frequencies, for one of the graph, are listed in Tab. 1. The corresponding graphlet frequency distribution is shown in Fig. 7.

Tab. 1. Frequencies of graphlets g_1, g_2, ..., g_9 (see Fig. 5) for the random geometric graph generated by the NetworkX (8000 vertices and 24087 edges).

Number of graphlet	1	2	3	4	5	6	7	8	9
Graphlet count	24087	59615	28564	159198	16525	1171	114511	39614	20453

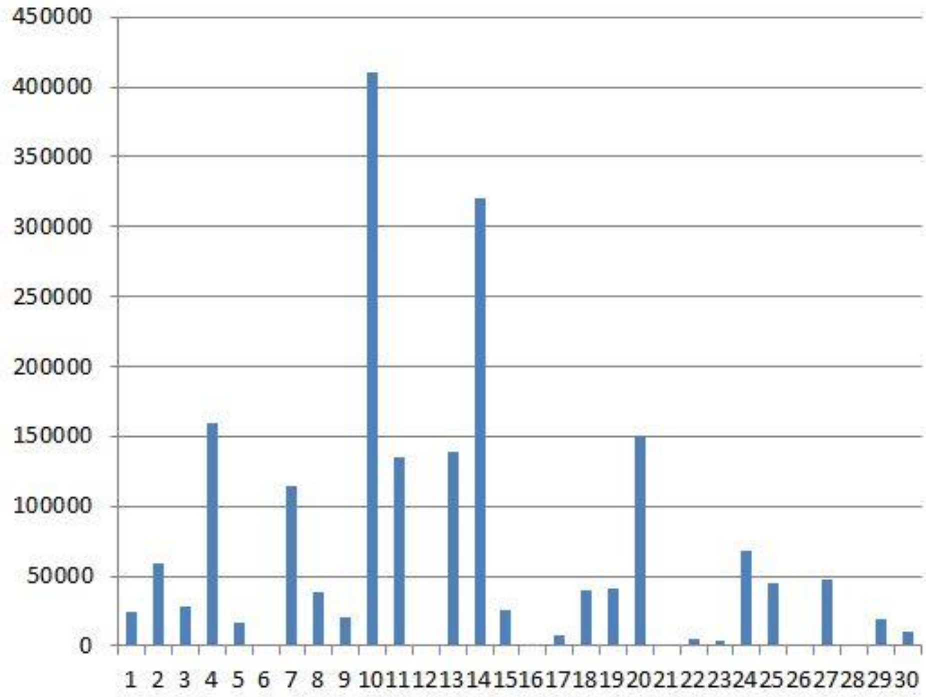

Fig. 7. Distribution of graphlet abolute counts in a random geometric graph (8000 vertices and r=0.01557). The graphlet number is on x-axe, the graphlet count (frequency) is on y-axe.

6 Conclusions

This chapter is focused on the frequent subgraph mining (FSM) problem and the related issues, i.e. the subgraph mining and the graphlet frequency counting, respectively. The main algorithmic principles of the FSM problem are explained. The authors of this chapter are actively doing research in the area of subgraph mining and the graphlet-based network analysis. Their research is motivated by the chemical structures and complexes and structure of PPI networks. Some details and research directions on the above topics are mentioned, as well.

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III

Measurement of the relative effectiveness of information channels for different groups of people based on data from statistical survey

Milan Terek

1 Introduction

The measurement is the assignment of a number to a characteristic of an object or event, which can be compared with other objects or events. The scope and application of measurement are dependent on the context and discipline[1]. In this chapter the measurement of the relative effectiveness of information channels for different groups of people based on data from statistical survey is studied. In general, the odds ratio is one from the measures of the association strength between two categorical variables. If the values of the first variable represent information channels and values of second one represent different groups of people obtaining information through these information channels, then the value of odds ratio can be understood and interpreted as a value of relative effectiveness of information channels. So the odds ratio has very specific interpretation in this context.

Frequently the same information is provided by more different information channels in the framework of one time period. For example a firm informs the people about offered services by paper publicity materials, internet and publicity in media. It is interested in the effectiveness of the information channels for the potential clients from different regions. The council of town informs the citizens about its activities by the network of billboards, its web page and communal media. It would like to know the effectiveness of information channels for different age categories of citizens.

Natural first step of such analysis is asking the people what information channel from they obtain or obviously obtain the information – to realize statistical survey. The questions concerning information channels using and different groups of people are generally categorical variables. The association between two categorical variables will be analyzed – categories of the first variable represent information channels and categories of second one represent

[1] For more details see https://en.wikipedia.org/wiki/Measurement.

different groups of people obtaining information through these information channels. The tree steps procedure of information channels effectiveness assessment for different groups of people will be suggested.

2 Analyzing Association between Two Categorical Variables

Categorical variables take values (categories) enabling to identify an attribute of each element. When only the identifying an attribute is possible, the measurement scale of the variable is nominal. When the values of variable exhibit the properties of nominal data and the order or rank of the values is meaningful, the variable is called ordinal – it is measured in ordinal scale[2].

When results of one variable tend to change as the results of the other variable take different values, we conclude there exists an association between those variables. Data for categorical variables association analysis are summarized in contingency tables. This type of table displays the number of elements observed at all combinations of possible outcomes for the two variables. The association between two nominal variables will be analyzed.

2.1 The Procedures of Association Analysis

Commonly three procedures of association analysis between two nominal variables are advised in the literature. The first consists of two steps – the realization of the statistical test revealing if the association between variables exists and if that is the case, the measurement how strong is the existing association with aid of some summary measures of association, such as Cramer`s *V*, contingency coefficient or Goodman and Kruskal's lambda[3].

The second procedure includes the testing of association and the using of adjusted standardized residuals enabling the study of the structure of association when the association was confirmed[4]. The adjusted standardized residuals serve for identification cells of contingency table which are "responsible" for revealed association.

The last procedure consists of association testing, the using of adjusted standardized residuals in the case of confirmed association and the measurement of the strength of

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[2] For more details about classification and measurement scales of variables see Anderson et al. (2007).
[3] See Dagnelie (1998), Miller & Miller (2004).
[4] See Sharpe et al. (2010).

association by odds ratios[5]. For information channels relative effectiveness measurement in the stated context, the last mentioned procedure will be useful.

2.1.1 Probability structure for contingency tables

Let both *X* and *Y* are response (dependent) variables, *X* with *r* categories and *Y* with *c* (*c* > 2) categories. A table having *r* rows and *c* columns for categories of *X* and *Y* have *rc* cells for possible outcomes. When the cells contain frequency counts of outcomes for a sample, the table is a contingency table. Let`s have a random sample of elements from a population. Then the responses (*X*, *Y*) of a randomly chosen elements have a probability distribution. Let π_{ij} be the probability that (*X*, *Y*) occurs in the cell in row *i* and column *j*. Then the probability distribution $\{\pi_{ij}\}$ is the joint probability distribution of *X* and *Y*. The row totals $\pi_{i.}$ and column totals $\pi_{.j}$ of joint probabilities define the marginal probability distributions.

Let *Y* is response variable and *X* is explanatory (independent) variable. For a fixed category of *X*, variable *Y* has a probability distribution. Let $\pi_{j|i}$ denotes the probability of classification of element in column *j* of *Y*, given that element is classified in row *i* of *X*. Then the probabilities $\{\pi_{1|i}, \pi_{2|i}, \ldots, \pi_{c|i}\}$ define the conditional probability distribution of *Y* at category *i* of *X*.

Similarly, the joint, marginal and conditional frequency distributions can be defined. Based on the data from random sample, the frequency distribution can be determined. The probabilities of a probability distribution can be estimated by relative frequencies of corresponding frequency distribution.

2.1.2 Chi-squared test of homogeneity

We speak of homogeneity in statistics when statistical characteristics of one part of a data set are the same as characteristics of another part of that data set. We will look at chi-squared test of homogeneity. Let *Y* is response variable and *X* is explanatory variable. The categories of *X* define *r* different populations. The random samples from *r* multinomial

[5] See Agresti & Finley (2014).

populations[6] with *c* different outcomes are sampled in that test[7]. In a contingency table, the sample sizes in the last column are fixed, the column totals are influenced by randomness of sampling.

We are testing:

H_0: The population conditional distributions are identical for all *r* populations, formally:

H_0: $\pi_{j|1} = \pi_{j|2} = \ldots = \pi_{j|r}$ for $j = 1, 2, \ldots, c$

meaning that random samples are from *r* populations with the same multinomial distribution versus an alternative

H_1: The population conditional distributions are not identical for all *r* populations, formally:

H_1: $\pi_{j|1}, \pi_{j|2}, \ldots, \pi_{j|r}$ are not all equal for at least one value of *j*

When a response variable is identified and the population conditional distributions are identical for all populations, they are said to be homogeneous (Agresti & Finlay 2014, p. 229). Let in contingency table n_{ij} be the observed frequency in *i*-th row and *j*-th column, $n_{i.}$ be the sum of n_{ij} values in the *i*-th row, $n_{.j}$ be total of n_{ij} values in the *j*-th column. The total of all n_{ij} values is *n*. Let o_{ij} denote an expected frequency. This is the count expected in a cell if the populations were homogeneous (H_0 is true). It equals the product of the row and column totals for that cell, divided by the total sample size. Formally, assuming H_0 is true, expected frequencies are calculated as follows:

$$o_{ij} = \frac{n_{i.} \cdot n_{.j}}{n} \quad (1)$$

[6] Multinomial distribution, see for example in Agresti (2013) or Freund (1992).
[7] For more details about random sampling from finite and infinite populations, see Anderson et al. (2007, pp. 227–229) or Terek (2017, pp. 107–110).

The value of test statistic is calculated according to following relationship:

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{\left(n_{ij} - o_{ij}\right)^2}{o_{ij}}$$

The critical region at level of significance *α* is $\chi^2 \geq \chi^2_{1-\alpha}\left((r-1)(c-1)\right)$ where $\chi^2_{1-\alpha}\left((r-1)(c-1)\right)$ is the (1 − *α*) quantile of chi-squared distribution with $(r-1)(c-1)$ degrees of freedom. When H_0 is true, n_{ij} and o_{ij} tend to be close for each cell, and χ^2 is relatively small. If H_0 is false, at least some n_{ij} and o_{ij} values tend not to be close, leading to a large test statistic. The larger value of test statistic, the greater evidence against H_0 − homogeneity.

Due to the considered test statistic following the chi-squared distribution only approximately, it is recommended to use this test only in the cases when no o_{ij} values are lower than 5. When variables are ordinal there are also another possibilities of association testing based on some ordinal characteristic of association, for example gamma, Kendall`s Tau-b, etc[8]. When *c* = 2, the problem leads to the test concerning differences among more proportions, based on binomial distribution[9].

2.1.3 Chi-squared test of independence

Let *X* and *Y* are both response variables. Then one sample from multinomial population with *rc* different outcomes is sampled. The statistical independence is tested and the test calls the test of independence[10].

We are testing:

[8] For more details about such characteristics see Agresti, 2010, pp. 188 – 190.

[9] For more details see Freund, 1992, p. 185.

[10] For more details about chi-squared test of homogeneity and independence see Agresti, 2013 or Miller & Miller, 2013.

H_0: The variables are statistically independent, formally:

H_0: $\pi_{ij} = \pi_{i.} \cdot \pi_{.j}$ for $i = 1, 2, \dots r$ and $j = 1, 2, \dots c$

versus an alternative

H_1: The variables are statistically dependent, formally:

H_1: $\pi_{ij} \neq \pi_{i.} \cdot \pi_{.j}$ for at least one pair of values of *i* and *j*,

where $\pi_{i.}$ is the marginal probability in row *i*, $\pi_{.j}$ is the marginal probability in column *j*.

Two categorical variables are statistically independent if the population conditional distributions on one of them are identical at each category of the other. The variables are statistically dependent if the population conditional distributions are not identical (Agresti & Finlay, 2014, p. 223). Statistical independence is a symmetric property between two variables: If the conditional distributions within rows are identical, then so are the conditional distributions within columns (Agresti & Finlay, 2014, p. 224). In the chi-squared test, the value of χ^2 test statistic does not depend on which is the response variable and which is the explanatory variable (if either). The steps of the test procedure and the results are identical either way (Agresti & Finlay, 2014, p. 229). The testing procedure is the same as it was mentioned above in the case of homogeneity, differences among more proportions (when required value of proportion is not specified) or statistical independence testing.

It is clear that the homogeneity of conditional distributions in a test of homogeneity imply statistical independence of corresponding variables. So, equivalent interpretations of a test of homogeneity results are possible. When we rejected the null hypothesis in a test of homogeneity, we can conclude that we obtained the evidence that the conditional distributions of response variable *Y* on *X* are not identical, or that variables *X* and *Y* are statistically dependent, or simply that *X* and *Y* are associated.

2.1.4 Residual Analysis

A cell-by-cell comparison of the observed and expected frequencies reveals the nature of the evidence about the association between variables. The difference $(n_{ij} - o_{ij})$ between an observed and expected cell frequency is called a residual. The adjusted standardized residuals for two nominal variables[11] can be defined as (Agresti & Finlay, 2014, p. 230):

$$r_{ij} = \frac{n_{ij} - o_{ij}}{\sqrt{o_{ij}\left(1 - \frac{n_{i.}}{n}\right)\left(1 - \frac{n_{.j}}{n}\right)}} \qquad \text{for } i = 1,2...r; \; j = 1, 2, ...c \qquad (2)$$

where $\frac{n_{i.}}{n}$ is an estimated marginal probability in row *i*,

$\frac{n_{.j}}{n}$ is an estimated marginal probability in column *j*.

The denominator in formula (2) is a standard error of random variable $(n_{ij} - o_{ij})$, when null hypothesis H_0 about statistical independence of variables is true. Adjusted standardized residuals r_{ij} follow asymptotically the standard normal distribution. They can be used to describe the pattern of the association among the table cells. A too large value of an adjusted standardized residual (greater than 2 in absolute value) indicates a deviation from independence in the cell.

2.1.5 Measures of Association Strength for Categorical Variables

A measure of the association strength is a statistic or parameter that indicates the strength of an association between two variables (Agresti & Finlay, 2014, p. 233). There are more summary measures of association strength between two nominal variables. Two approaches to summarize the strength of the association between nominal variables are known:

- Coefficients based on the χ^2
- Coefficients based on proportional reduction of prediction error (PRE)

[11] For ordinal variables see Agresti, 2010, p. 73 – 74.

Association Coefficients Based on the Chi-Squared

The χ^2 statistic as such is not used to measure association between two variables, but it serves as the element in association coefficients construction. The following association coefficients based on χ^2 are suggested in the literature: Phi–squared (φ^2), Cramer's *V* and contingency coefficient.

The Cramer`s *V* is most frequently used. It is defined as

$$V = \sqrt{\frac{\chi^2}{n \cdot h}},$$

where *h* is the minimum from (r – 1) and (c – 1). Cramér's *V* varies from 0 (corresponding to no association between the variables) to 1 (complete association) and can reach 1 only when the two variables are equal to each other.

Association Coefficients Based on PRE

Goodman and Kruskal introduced the idea of proportional reduction in error of prediction. Two association coefficients based on this idea are known − the Goodman and Kruskal's lambda and Goodman and Kruskal's tau.

The Goodman and Kruskal's lambda (λ) measures the percentage improvement in predictability of the response variable (row variable or column variable), given the value of the other variable (column variable or row variable). The value of λ_r for row response variable is

$$\lambda_r = \frac{\sum_i \max n_{ij} - \max(n_{.j})}{n - \max(n_{.j})}$$

The value of λ_c for column response variable is

$$\lambda_c = \frac{\sum_j \max n_{ij} - \max(n_{i.})}{n - \max(n_{i.})}$$

The symmetric (non-directional) lambda (λ) can be also calculated. It lies between the values of λ_r and λ_c . The symmetric lambda is defined as

$$\lambda = \frac{\sum_i \max n_{ij} + \sum_j \max n_{ij} - \max(n_{i.}) - \max(n_{.j})}{2n - \max(n_{i.}) - \max(n_{.j})}$$

Asymmetric and symmetric lambda take values from the interval [0, 1]. The information obtained by summary measures of association is interesting but no so useful for managing the information flows.

Another Measures of Association Strength

When 2 x 2 contingency table is analyzed, the difference of proportions can be used as measure of association.

Odds Ratio

The odds ratio is the measure of association that can be used in all contingency tables. We will use success to denote the outcome of interest and failure the other outcome. For a response variable with two values, the odds for success is defined as:

$$\text{Odds} = \frac{\text{Probabilit y of success}}{\text{Probabilit y of failure}}$$

The estimated odds (calculated value is only estimate of the real unknown value of odds in the population that is why it is called estimated) for a response variable with two values equals the number of successes divided by the number of failures. The odds ratio θ in 2 x 2 contingency tables equals to the ratio of the 1st row odds to the 2nd row odds. In *r x c* contingency tables, odds ratio can be calculated in any 2 x 2 sub-table. The odds ratios will be used as the measure of relative effectiveness of information channels in the framework of stated problem.

The situation is different in the case of two ordinal variables. For *r* x *c* tables, odds ratios can use each pair of rows in combination with each pair of columns. The local, global and cumulative odds ratios are defined[12]. When variables are ordinal, the local odds ratios can be used for relative effectiveness of information channels measurement in the framework of stated problem.

3 The Study of Information Channels about Academic Ethics

Effectiveness

A statistical study aimed to obtaining information about academic ethics program effectiveness at one Slovak university – School of Management in Trencin/City University of Seattle, was realized. There were considered many groups of students at university − according to place of study, mode of study (online, in-class), degree of study (bachelor, magister), language of study (Slovak, English) and the year of bachelor study (first, second and third). Each group defines one population. The statistical survey was realized. The students from each population were sampled by random sampling with replacement. The questionnaire contained questions related with academic ethics program at the university. The answers to the questions associated with diverse school branches, diverse degree of study, diverse modes of study, diverse year-class, diverse languages of study and diverse gender of students were analyzed. It was examined how students perceive functioning of the program, what are the preferred information channels as well as differences among diverse specific groups of students[13]. The question number 3 is: "How did you learn about academic ethics rules and procedures at the university?" The answers to this question related with two groups of students − studying in Slovak and studying in English will be analyzed[14]. Originally there were 8 possible information channels. The joining of the columns with expected frequencies less than 5 was realized. Three

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[12] For more details see Agresti, 2010, pp. 14 – 22.
[13] For more details see Krocity, 2015.
[14] See also Terek, 2016.

different information channels remaining after joining are presented in Table 1. So finally we have two nominal variables – “Information channels” with three categories and “Groups of students” with two categories. The table 1 includes expected frequencies (in parentheses) calculated according to (1).

Table 1 Distribution of Information Channels at Categories of Slovak and English Language Programs

Information Channels / **Language of study**	1st Information Channel (New students orientation session)	2nd Information Channel (Teacher)	3rd Information Channel (Class syllabus, www.vsm.sk, University Catalogue, Classmates, Noticeboards, Other sources)	$n_{i.}$
Slovak language	2 (6.97)	4 (5.36)	61 (54.67)	67
English language	11 (6.03)	6 (4.64)	41 (47.33)	58
$n_{.j}$	13	10	102	125

Source: Developed by author

Decision Making about Association between Information Channels and Groups of Students

We are interested in whether the distribution of information channels for different groups of students is the same. Formally, there are two categorical variables – „Information channel“ with three categories and „group of students“ with two categories. The values of the first variable will define the possible outcomes, the values of second variable will define two populations – students of Slovak language program and students of English language program. The first variable is identified as response variable, second one as explanatory variable.

Let` s see why the rule (1) make sense. In the example, $\frac{n_{.1}}{n}$ is the proportion of students of both study programs using first information channel. If the conditional distributions were identical, we would expect the same proportion of students using the first information channel in each of study programs. Then the number of students using this information channel in the students of Slovak language program group would be $\frac{n_{.1}}{n} \cdot n_{1.}$ and in the the students of English language program group: $\frac{n_{.1}}{n} \cdot n_{2.}$. We can see that expected frequencies were calculated according to (1).

The obtained p-value was 0.006862. Thus we can reject the null hypothesis in favor of the alternative hypothesis. We conclude that random sample from the Slovak language program student`s population and the random sample from the English language program student`s population does not come from the same probability distribution. There is an association between the information channels about academic ethics using and the language of study. In other words, the information channels about academic ethics effectiveness differs by the language of study.

Revealing the Information Channels Relative Effectiveness by Residual Analysis

The revealing of information channels relative effectiveness was realized by residual analysis. The adjusted standardized residuals were calculated according to (2), to find the cells "responsible" for the association. The results are in Table 2.

There is considerably great positive value of residual concerning students studying in English language obtaining information about academic ethics through the first information channel − at new students orientation sessions in the Table 2. For students studying in Slovak language there is a considerably great negative value of residual in the same column. This means that there are more students studying in English who obtain information through the first information channel than it is suggested by the homogeneity hypothesis. The great negative residual in the same column for students studying in Slovak means that there are less students studying in Slovak who obtain information through the first information channel than it is suggested by the homogeneity hypothesis.

Table 2 − Information Channels and Adjusted Standardized Residuals for Different Language of Study

Information Channels / **Language of study**	1st Information Channel (New students orientation session)	2nd Information Channel (Teacher)	3rd Information Channel (Class syllabus, www.vsm.sk, University Catalogue, Classmates, Noticeboards, Other sources)
Slovak language	− 2.92	− 0.9	2.93
English language	2.92	0.9	− 2.93

Source: Developed by author

The great positive value of residual in the fourth column concerning students studying in Slovak indicates that there are more students studying in Slovak who obtain information through the third information channel than it is suggested by the homogeneity hypothesis. The great negative residual in the same column for students studying in English means that there are less students studying in English who obtain information through the third information channel than it is suggested by the homogeneity hypothesis.

We can conclude that students studying in English are more likely to obtain the information through the first information channel − at new students orientation sessions than students studying in Slovak. The students studying in Slovak are more likely to obtain the information through the third information channel than students studying in English. In other words the first information channel is relative more effective for students studying in English than for students studying in Slovak. The third information channel is relative more effective for students studying in Slovak than for students studying in English.

It is also clear from Table 2 that there is no important difference between effectiveness of second information channel for the students studying in Slovak and in English. That means also that among Slovak and English program teachers is not difference in providing information about academic ethics. This is also very useful information for information channels assessment and finally for information flows managing.

Information Channels Relative Effectiveness Measurement

Odds ratio analysis will be applied. In general, an arbitrary 2 *x* 2 sub-table can be analyzed by the odds ratio. This analysis enables to take in consideration all cells of contingency table where association was revealed by residual analysis. The sub-table with the second and forth columns of Table 1 will be analyzed. The data are in Table 3. The association strength measured by odds ratio will be understood and interpreted as value of relative effectiveness.

Table 3 Sub-table of Table 1

Information Channels / **Language of study**	1st Information Channel (New students orientation session)	3rd Information Channel (Class syllabus, www.vsm.sk, University Catalogue, Classmates, Noticeboards, Other sources)	Total
Slovak language	2	61	63
English language	11	41	52

Source: Developed by author

The relative effectiveness measurement of the first information channel for the students studying in English in comparison to the students studying in Slovak will be realized. The first information channel (second column) will represent a success and the third one (third column) a failure.

The estimated odds for students studying in English is

$$\frac{\frac{11}{52}}{\frac{41}{52}} = \frac{11}{41} \approx 0.268$$

As far as students studying in English are concerned, there are about 0.268 of a student who obtained information through the first information channel per 1 student who obtained the information through the third information channel.

The estimated odds for students studying in Slovak is

$$\frac{\frac{2}{63}}{\frac{61}{63}} = \frac{2}{61} \approx 0.033$$

As far as students studying in Slovak are concerned, there are about 0.033 of a student who obtained information through the first information channel per 1 student who obtained the information through the third information channel.

The odds ratio for students studying in English and for students studying in Slovak can be calculated as follows

$$\theta = \frac{0.268}{0.033} \approx 8.121$$

The student studying in English has 8.121 times greater chance to obtain information through the first information channel − at the new students orientation session than a student studying in Slovak language. This value can be considered as the assessment of relative effectiveness of the first information channel for the students studying in English in comparison to the students studying in Slovak. So the first information channel is relatively 8.121 times more effective for students studying in English than for students studying in Slovak.

It can be easily shown that when the third information channel represents success and the first one, failure, the same odds ratio for the students studying in Slovak will be obtained. The student studying in Slovak has 8.121 times greater chance to obtain information through the third information channel than a student studying in English language. So the third information channel is relatively 8.121 times more effective for students studying in Slovak than for students studying in English. The analysis of the first and second and the second and third information channels can be realized by the same way.

4 The Effectiveness Evaluation of Information Channels about Firm`s Services

A firm carried out a statistical survey focused on customer satisfaction and their profiles study. The 400 customers were randomly selected and asked to fill in a questionnaire. One of the closed questions in the questionnaire was: "From which information source did you obtain the first information about the services provided by our firm?" The answer distribution is shown in Table 4 (in parentheses there are the expected frequencies)[15].

The Pearson chi-squared test of independence offered *p*-value equal to 0.000106. That means that H_0 can be rejected in favor of the alternative hypothesis. We can conclude that variables „information channels" and „permanent residence of customer" are statistically dependent. The effectiveness of the information channels about services provided by the firm differs by the permanent residence of a customer.

Table 4 Distribution of Information Channels at Categories of Customer`s Permanent Residence

Information Channels Permanent residence of customer	1st Information Channel (paper publicity materials)	2nd Information Channel (internet)	3rd Information Channel (publicity in media or other sources)	$n_{i.}$
Bratislava city	150 (134.375)	74 (74.375)	26 (41.250)	250
Region Bratislava (except Bratislava city)	20 (31.713)	21 (17.553)	18 (9.735)	59
Elsewhere	45 (48.913)	24 (27.073)	22 (15.015)	91
$n_{.j}$	215	119	66	400

Source: Developed by author

[15] See also Terek, 2018.

Residual Analysis and Information Channels Relative Effectiveness

The adjusted standardized residuals were calculated according to (1), to find the cells causing the association. The results are in Table 5.

Table 5 Information Channels and Adjusted Standardized Residuals for the Different Permanent Residence of the Customer

Information Channels / **Permanent residence of customer**	1st Information Channel (paper publicity materials)	2nd Information Channel (internet)	3rd Information Channel (publicity in media or other sources)
Bratislava city	3.24	− 0.08	− 4.24
Region Bratislava (except Bratislava city)	− 3.31	1.06	3.14
Elsewhere	− 0.94	− 0.80	2.24

Source: Developed by author

Based on results in Table 5, we can conclude that there are more customers from the city Bratislava and fewer customers from the region of Bratislava who obtained information through the first information channel than it is suggested by the independency hypothesis. There are more customers from the region of Bratislava and fewer customers from the city Bratislava who obtained information through the third information channel than it is suggested by the independency hypothesis. We can conclude that the first information channel is relatively more effective for customers from the city Bratislava than for customers from the region of Bratislava and the third information channel is relatively more effective for customers from the region of Bratislava than for customers from the Bratislava city.

Relative Effectiveness Measurement of Information Channels

The odds ratio analysis will be applied. The association strength measured by odds ratio will be understood and interpreted as a value of relative effectiveness. The only appropriate is to analyze the sub-tables for which the corresponding residuals are greater than 2 in the absolute value. We will analyze the following sub-table (in Table 6).

Table 6 Sub-table of Table 4

Information Channels / **Permanent residence of customer**	1st Information Channel (paper publicity materials)	3rd Information Channel (publicity in media or other sources)	Total
Bratislava city	150	26	176
Region Bratislava (except Bratislava city)	20	18	38

Source: Developed by author

The relative effectiveness assessment of the first information channel for the customers from the city Bratislava in comparison to the customers from the region of Bratislava will be realized. The first information channel (the second column) will represent a success and the third one (the third column) will represent a failure.

The estimated odds for customers from the Bratislava city is

$$\frac{\frac{150}{176}}{\frac{26}{176}} = \frac{150}{26} \approx 5.7692$$

There are about 5.7692 of customers from the Bratislava city who obtained information through the first information channel per 1 customer who obtained the information through the third information channel.

The estimated odds for customers from the region of Bratislava is

$$\frac{\frac{20}{38}}{\frac{18}{38}} = \frac{20}{18} \approx 1.1111$$

There are about 1.1111 of customers from the region of Bratislava who obtained information through the first information channel per 1 customer who obtained the information through the third one.

The odds ratio for customers from the Bratislava city and for customers from the region of Bratislava can be calculated as follows:

$$\theta = \frac{5.7692}{1.1111} \approx 5.1923$$

A customer from the Bratislava city has a 5.1923 times greater chance to obtain information through the first information channel than a customer from the region of Bratislava. The first information channel is relatively 5.1923 times more effective for customers from the Bratislava city than for customers from the region of Bratislava.

Also in this case, when the third information channel represents success and the first one failure, the same odds ratio for the customers from the region of Bratislava will be obtained. This relation concerning odds ratios in contingency sub-tables is generally valid.

5 Procedure of Information Channels Effectiveness

Measurement

The different information channels are expressed by the *c* values of the first categorical variable *Y*, the different groups of people obtaining information through these information channels are expressed by the *r* values of the second categorical variable *X*.

When *Y* is response variable and *X* is explanatory variable, the categories of *X* define *r* different populations. Then the random samples from *r* multinomial populations with *c* different outcomes are sampled and the homogeneity is tested by chi-squared test in the first step of the procedure.

When *X* and *Y* are both response variables, then one sample from multinomial population with *rc* different outcomes is sampled and the statistical independence is tested by chi-squared test in the first step of the procedure. The same testing procedure is applied in the both cases.

Observed frequencies n_{ij} are obtained on the basis of answers of respondents sampled by random sampling with replacement from each from *r* populations in the case of homogeneity testing or from one population in the case of independence testing. The respondents indicate the information channel they obtained information through.

Then the testing of homogeneity or independence is applied. When the null hypothesis is not rejected, we cannot conclude that the effectiveness of information channels is significantly different for different groups of people. When the null hypothesis is rejected in favor of alternative hypothesis, we conclude that the effectiveness of information channels for different groups of people is not identic. Then the residual analysis is recommended.

Residual analysis with aid of adjusted standardized residuals determines the cells of contingency table "causing" association. The residuals greater than 2 in absolute value indicate the cells "responsible" for association. When $r_{ij} > 2$, the corresponding information channel is more effective for corresponding group of people as for other groups, when $r_{ij} < -2$, the corresponding information channel is less effective for corresponding group of people as for other groups.

Finally, information channels relative effectiveness measurement based on the odds ratios[16] is recommended:

[16] Local odds ratios in the case of ordinal variables.

- Each 2 *x* 2 sub-table to be analyzed contains at least one cell with $|r_{ij}| > 2$.
- The estimated odds are calculated for the first channel from the left as success and for the second channel from the left as failure. Then the odds ratio can be calculated with the greater odds in numerator. The result is the odds ratio for the group with greater value of odds.
- The calculated odds ratio is the same for the second channel from the left as success and for the first channel from the left as failure, for the group with less value of odds.
- The odds ratio is understood and interpreted as value of information channels relative effectiveness.

6 Conclusions

It was showed how to use the data from statistical survey for the measurement the relative effectiveness of information channels for different groups of people. The testing of association enables to make decision about whether there exists an association between variables one of which represents information channels and second one, different groups of people obtaining information through these information channels. In general if the samples from *r* different populations given by different groups of people is realized, the homogeneity is tested, if only one sample from one population is realized, the independence is tested. In both cases the association between two variables is tested by the same testing procedure. The difference is only in the number of populations, which the random sampling with replacement is realized from.

When the null hypothesis is not rejected, the evidence about an association between variables was not obtained and we cannot conclude that there is a difference in relative effectiveness of information channels for different groups of people. When the null hypothesis was rejected, we can conclude that there is an association between variables. The evidence about a difference in relative effectiveness of information channels for different groups of people was obtained.

Once an association between variables is established, obviously some summary measures of association strength are calculated. The Cramer`s *V* and Goodman and Kruskal's lambdas were described. It can be shown that these measures are not very useful in the information flows management. It is necessary to analyze which combinations of variable values cause the identified association. Then the using of residual analysis is recommended. This statistical tool enables to determine which combinations of variables categories cause the revealed association. The identification of cells "responsible" for association in a contingency table enables to recognize the relative effectiveness of information channels for different groups of people. When there is a great positive value of the adjusted standardized residual in the cell, corresponding information channel is more effective for the corresponding group of people. When there is a great negative value of the adjusted standardized residual in the cell, corresponding information channel is less effective for the corresponding group of people.

The using of odds ratios is recommended for information channels relative effectiveness measurement. This characteristic can be calculated for all more effective information channels revealed by residual analysis.

In the application of the proposed procedure in the first study, the results that the first information channel is relatively 8.121 times more effective for students studying in English than for students studying in Slovak and the third information channel is relatively 8.121 times more effective for students studying in Slovak than for students studying in English were obtained. In the context of information flows management in the area of academic ethics at the university, one of the most important objectives is to provide the academic ethics rules to all students as soon as possible that means at the beginning of studies, for example during new students orientation session (it is the first information channel). The results of the analysis show that whereas the new student orientation session is an effective information channel for students studying in English language, it is not for students studying in Slovak language. It is clear that new student orientation session is visited more by students studying in English language but no so frequently by students studying in Slovak language. So some modification of existing or creation some new, more effective information channels for students studying in Slovak language can be recommended. The realized statistical study provided naturally also another interesting results utilizable mainly in the process of information flows control at the mentioned university, but the applied methodology could be useful also for other universities.

In the application of the proposed procedure in the second study, the results showing that the first information channel is relatively 5.19 times more effective for customers from the Bratislava city than for customers from the region of Bratislava and that the third information channel is relatively 5.19 times more effective for customers from the region of Bratislava than for customers from the city of Bratislava were obtained. That could be very useful information for managing the information channels using in different regions.

Described procedure of analysis can be used also in a lot of others contexts. For example, it is current practice to ask the hotel guests to fill the questionnaires with questions one of which is frequently “how did you learn about our hotel?” Each gest has to show in hotel reception desk identity card where are names, date of birth, domicile and so on. In this case the testing of independence could be realized. The similar analysis as described could be realized on the basis of the data from survey by hotel management. Obtained information could be very useful for example in some marketing activities. The using of some specialized statistical software is not required in order to perform the analysis, Excel is fully sufficient.

The described procedures could be used not only for the information channels effectiveness assessment. For example for a hotel management could be interesting the association between satisfaction with services in the hotel and different groups of guests and so on.

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IV

Analyzing and Modeling Wage Distributions: a Case Study of the Czech Republic

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1 Introduction

A wage is monetary compensation paid by an employer to an employee in exchange for work done. Wage rates are in general influenced by market forces (supply and demand) and by the legislation of the respective country. Further factors influencing wages on a personal level are sex, age, and education. Statistical analysis of the development of the wage and income distribution is a crucial precondition for economic modeling of the labor market processes.

There are two ways how to analyze wage data. The first descriptive will allow us to evaluate wage developments over time for the whole Czech Republic taking into account various factors. The second, analytical approach, makes it possible to perform probabilistic analyses and to create probabilistic models of the wage distribution. We will use both approaches in our text. We start with descriptive analysis and use probabilistic analysis at the end of the text.

The goal of the text is to analyze and compare wages in the Czech Republic. We start our work with the average wage. We analyze its evolution, growth rate and trend. Since the average value is far from being an ideal characteristic of wages, certain quantiles, in particular, the median, will also be analyzed. We will also observe the empirical distribution of wage frequencies and the time evolution of that distribution. The changes in the wage distribution will provide us with a clear idea of how the basic characteristics of the wages – the average value, variability, skewness, and kurtosis – have been changing. A detailed comparison by gender, as well as by age and other factors, will then be carried out. The wage inequality levels will also be observed, as well as the time evolution of these levels over the last years. We will employ the Gini index for this purpose. But not only the analysis of wage distributions is presented in the text. We use various probabilistic distributions to model the empirical wage distribution and we use these models to predict the future wage distributions.

2 Data

Wage data for the Czech Republic (CR) for the 1995-2017 period is at our disposal. The source of our data is the consultant company Trexima (http://www.trexima.cz); this company collects the wage data for the Czech Ministry of Labor and Social Affairs, and the Czech Statistical Office (http://czso.cz) also makes use of this data. More information on the methodology of the wage data collection can be found in (Marek, Doucek, 2016).

The amount of data gradually increases from the sample size of about 300 000 in 1995 to more than two million in 2017. We deal with the basic statistical characteristics, out of which the average value of wages is the most frequently monitored one. The other characteristics include quantiles, namely, the first decile (10% quantile, D1), lower quartile (25% quantile, Q1), median (50% quantile), third quartile (75% quantile, Q3), and ninth decile (90% quantile, D9); the standard deviation, volume of working hours, and the size of the sample. The size of the sample in respective years is more than sufficient for the results of our analyses to be reliable and representative. For the calculations based on gender, age and other factors the sample sizes will, of course, be smaller, but sum up to the above-mentioned sample size. The data is always taken for the second calendar quarter in each year because of the long-term stability of the working hours' volume in that quarter. However, this fact may cause certain discrepancies between our values and the officially published statistics, which are based on the data of the entire calendar years. On the other hand, such discrepancies are small and do not distort the results. Analyses based on the data from previous years can be found, e.g., in (Marek, 2010). In addition to the basic characteristics, we also have, for each of the processed years, a detailed interval distribution of the data, where each interval's width is 500 CZK. Such a detailed data structure enables us to calculate other characteristics (such as additional quantiles, winsorized and truncated average values, etc.), see (Marek, 2018a) or (Terek, 2016).

3 Methodology

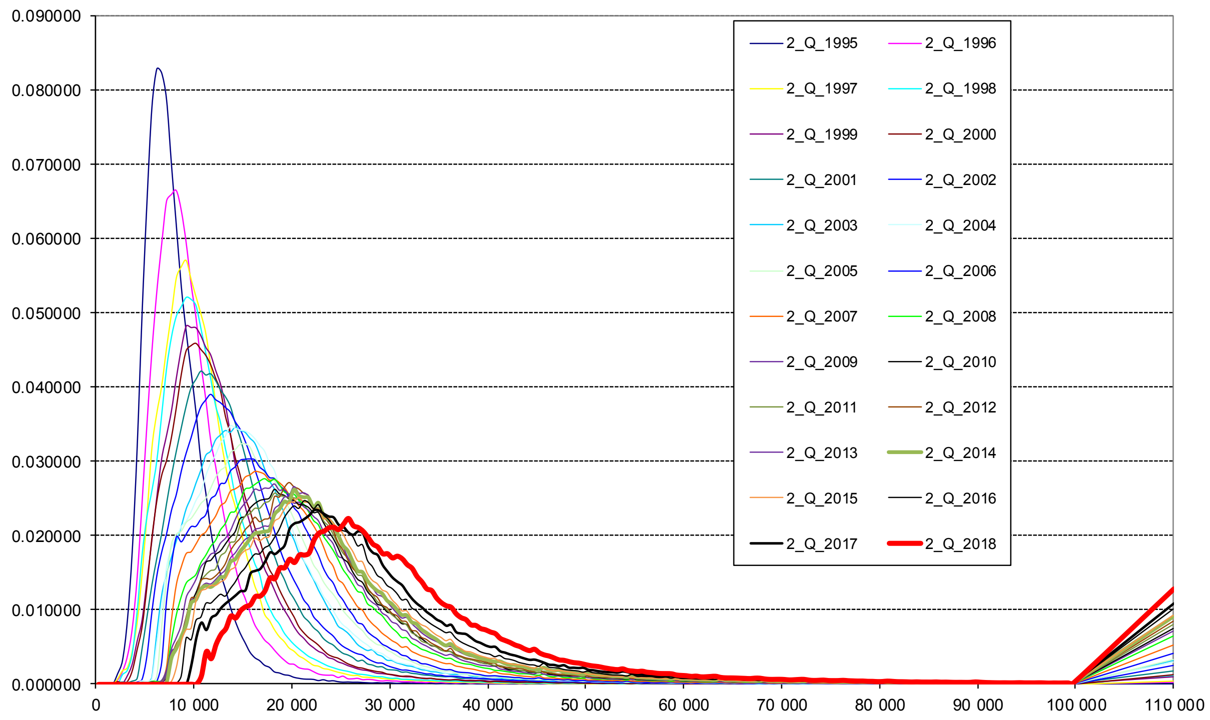
The wage data is given in the current prices of the respective years. The data is expressed in CZK. The usual sample statistical characteristics have been used as our comparison tool. Such characteristics have the character of annual time series; hence it makes sense to consider modeling the trends and calculate some other characteristics, such as the annual growth coefficient or the average growth coefficient for the entire considered period.

All of our analyses are executed in MS Excel. The main types of outcome are tables and charts, and we have used the *Data Analysis* procedure for purposes of analyzing the trends. We have used the usual method for calculating the values of the Gini index (Gini, 1912), based on the empirical Lorenz Curve derived from the relative-frequency table and smoothed by a 5th-degree polynomial. This high degree of the polynomial has been chosen to achieve a very good fit between the empirical data and the theoretical Lorenz Curve. The quality of the model is sufficient, with the value of the index of determination at 0.9996. The particular coefficients of the polynomial enabled us to calculate its integral between the limits 0 and 1, by which the area below the polynomial curve has been established. From that value, the Gini index was easy to derive.

4 Empirical wage distribution

First of all, we study the average values of wages over the years 1995-2018. Having at our disposal the interval distribution of the wage frequencies, we are able to study the polygon of the wage distribution, i.e., the empirical distribution of the wage frequencies, which enables us to identify a probabilistic model for the wage distribution (Marek, Vrabec, 2013), (Bartošová, Longford, 2014), and (Malá, 2015). Figure 1 shows the evolution of wages in CR.

Figure 1: Empirical wage distribution in CR



Source: Own graph

The wage distributions are presented in the same scale to facilitate their comparability. The depicted curves are non-smooth because we deal with empirical data, not theoretical models. These Figures show that:

- the wage distribution has been changing – the location has been moving to the right (the wages have been growing), the variability has been increasing (the wage values have been more and more differing from the average and from each other) the skewness has been growing and the kurtosis decreasing;
- the proportion of high wages (the wages above 100,000 CZK, i.e. approx. 4,000 EUR) has been growing, i.e. the distribution's tail is growing;

The 24-year evolution of the wage distributions is analyzed in more detail in other text, where are illustrated all the changes we are describing above.

4.1 Average Wage

The average value of wages is one of the most frequently monitored and published economic indices. At the same time the average is far from being an ideal measure of the wage level. If the sample contains outliers (and this is also true for our data, as we will see below), the average value responds very sensitively to such outliers. Namely, for the wage data, it is often claimed that the average value is about a 67% quantile; the authors of this paper can support such an opinion by calculations (below in text). This statement means that about two-thirds of employees do not get the average wage. Hence it is recommendable that some other measures for the wage levels should be published; among those, quantiles are most suitable. The median and lower and upper quartiles, or possibly the first and last deciles are quantiles most often referred to. Some other characteristics presented later in the text could also be used, see (Marek, 2018b).

Table 1: Average wage and wage quantiles

Year	No.obs.	Average	StanDev	D1	Q1	Median	Q3	D9
1995	321,277	8,311	4,133	4,879	5,963	7,500	9,691	12,314
1996	405,138	9,962	5,393	5,645	7,047	8,956	11,505	14,748
1997	622,505	11,322	6,490	6,178	7,910	10,171	13,083	16,774
1998	953,691	12,026	8,261	6,287	8,114	10,563	13,801	17,911
1999	1,024,898	12,982	8,262	6,894	8,859	11,506	14,911	19,499
2000	1,053,536	13,541	9,651	6,981	9,077	11,860	15,570	20,435
2001	1,075,875	14,743	10,372	7,693	9,870	12,901	16,794	22,234
2002	1,107,991	15,964	12,994	8,181	10,564	13,857	18,058	24,003
2003	1,230,282	17,748	13,504	9,143	11,829	15,519	20,070	26,271
2004	1,680,800	17,759	13,062	9,185	12,073	15,789	20,168	26,143
2005	1,818,369	18,640	13,796	9,371	12,403	16,432	21,376	27,754
2006	1,976,571	19,526	17,696	9,710	12,882	17,143	22,192	28,828
2007	2,059,416	20,953	18,055	10,381	13,659	18,185	23,602	31,257
2008	2,079,765	22,338	20,714	11,060	14,583	19,267	25,094	33,306
2009	1,933,772	23,418	19,014	11,681	15,339	20,138	26,241	35,093
2010	1,956,702	24,077	19,316	12,084	15,778	20,753	27,009	36,143
2011	1,973,468	24,484	24,802	12,199	15,996	21,020	27,225	36,677
2012	1,999,934	24,829	20,109	12,255	16,281	21,319	27,583	37,328
2013	2,015,903	25,448	20,564	12,416	16,595	21,779	28,322	38,598
2014	2,056,133	25,728	19,612	12,570	16,821	22,074	28,794	39,182
2015	2,098,854	26,369	19,903	12,978	17,290	22,658	29,566	40,162
2016	2,119,396	27,668	20,478	13,944	18,391	23,757	30,963	42,026
2017	2,185,573	29,166	20,749	14,982	19,547	25,135	32,610	44,334
2018	2,237,108	31,992	21,576	16,926	21,714	27,738	35,878	48,137

Source: Trexima

The columns have the following meaning:

No.obs. – number of observations,

Average – average wage,

StandDev – standard deviation,

D1 – the first decile (a 10% quantile),

Q1 – lower quartile (a 25% quantile),

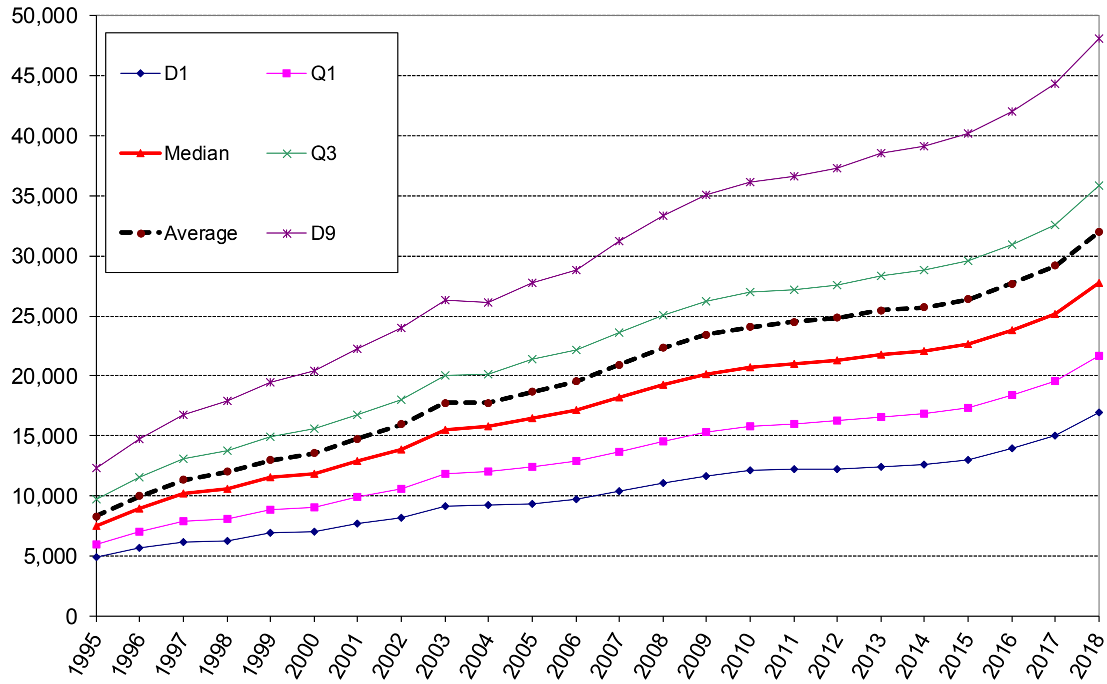
Median – a 50% quantile,

Q3 – upper quartile (a 75% quantile),

D9 – the ninth decile (a 90% quantile).

We showed these characteristics in a graph to see better their trend. The graph does not include the standard deviation of wages since it is a variability measure while other characteristics are location measures.

Figure 3: Wage characteristics



Source: Own graph

The data in the graph are in compliance with the legend as to the order from the top to the bottom. D9 is the highest value, after that, it is Q1, etc. Both the table and the graph show several general facts:

- wages keep going up over time linearly;
- the wage growth did not really slow down even during the economic crisis;
- the scissors of individual characteristics widen over time – it means, among other things, that the wage variability keeps growing, which is also obvious from the growing standard deviation;
- the ninth wage decile shows the fastest growth, which means that high wages keep growing much faster than other wages.

Table 2 provides an interesting view of the trend in quantile characteristics of wages and their mutual relationship.

Table 2: Differences and ratios of quantile measures

year	D9-D1	Q3-Q1	D9/D1	Q3/Q1	D9-Median	Median-D1	Q3-Median	Median-Q1	Average-Median
1995	7,436	3,728	2.52	1.63	4,815	2,621	2,192	1,536	811
1996	9,104	4,458	2.61	1.63	5,792	3,311	2,549	1,909	1,006
1997	10,596	5,173	2.72	1.65	6,603	3,994	2,912	2,261	1,151
1998	11,624	5,687	2.85	1.70	7,348	4,276	3,238	2,449	1,463
1999	12,606	6,052	2.83	1.68	7,994	4,612	3,405	2,647	1,476
2000	13,454	6,493	2.93	1.72	8,575	4,879	3,710	2,783	1,681
2001	14,541	6,924	2.89	1.70	9,333	5,208	3,893	3,030	1,842
2002	15,822	7,494	2.93	1.71	10,147	5,675	4,201	3,293	2,107
2003	17,128	8,241	2.87	1.70	10,751	6,377	4,551	3,690	2,229
2004	16,957	8,095	2.85	1.67	10,354	6,603	4,379	3,716	1,970
2005	18,383	8,973	2.96	1.72	11,322	7,061	4,944	4,029	2,208
2006	19,118	9,310	2.97	1.72	11,686	7,432	5,049	4,261	2,383
2007	20,876	9,943	3.01	1.73	13,072	7,804	5,417	4,526	2,768
2008	22,246	10,510	3.01	1.72	14,040	8,207	5,827	4,683	3,071
2009	23,412	10,902	3.00	1.71	14,954	8,458	6,103	4,799	3,280
2010	24,059	11,231	2.99	1.71	15,389	8,669	6,255	4,975	3,324
2011	24,477	11,229	3.01	1.70	15,656	8,821	6,205	5,024	3,464
2012	25,073	11,302	3.05	1.69	16,009	9,064	6,264	5,038	3,510
2013	26,182	11,727	3.11	1.71	16,819	9,363	6,543	5,184	3,669
2014	26,612	11,973	3.12	1.71	17,108	9,504	6,720	5,254	3,654
2015	27,185	12,276	3.09	1.71	17,504	9,681	6,908	5,368	3,711
2016	28,083	12,572	3.01	1.68	18,270	9,813	7,206	5,366	3,911
2017	29,351	13,063	2.96	1.67	19,199	10,152	7,476	5,588	4,031
2018	31,211	14,165	2.84	1.65	20,399	10,812	8,140	6,025	4,254

Source: Own calculations

Wage growth is not the same in all periods. While wages are rising steadily at current prices, growth rates are changing over time. Significant growth can be observed especially in the last years.

The time series format of our data enables us to calculate simple measures for their dynamics. Let us consider a time series with n observations $y_1, y_2 ..., y_n$. We will evaluate its time evolution in both absolute and relative terms (Cyhelský 1981). In particular, we calculate the average absolute increment

$$\bar{\Delta} = \frac{y_n - y_1}{n-1}, \quad (1)$$

and the average growth coefficient

$$\bar{k} = \sqrt[n-1]{\frac{y_n}{y_1}}, \tag{2}$$

Let's see what the average absolute increment and the average growth coefficient are.

Table 3: Basic characteristics of wage development

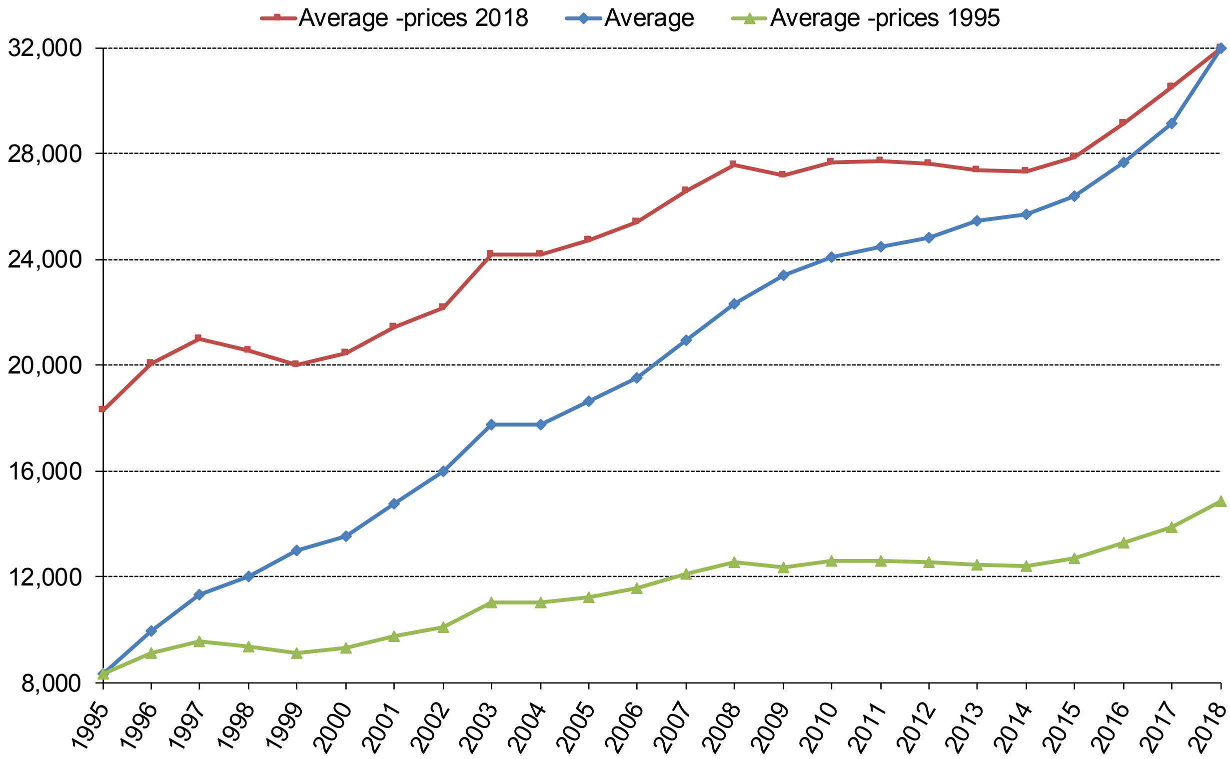
$\bar{\Delta}$	$\bar{k}$	Δ_{2018}	k_{2018}
987	1,06	2825	1,10

Source: Own calculations

Thus, wages have been growing by an average of 6% per year on a long-term basis, which totals CZK 987. It can be seen that absolute growth and its value have increased considerably in the last year of monitoring. An increase of 10% is absolutely unprecedented in the last years, only the values of 1996 and 1997 exceed it.

First of all, we compare the average values of wages over years. As the inflation rate evolution has been different in each year, we have re-calculated the average values to the prices valid at the end of 2018 and these re-calculated values will be used. To illustrate inflation's effects on the wages, the following Figure 4 shows the development of the average wage both in current prices, then in 1995 prices and finally in 2018 prices. Taking into account inflation, it can be seen that from the rising wage (middle curve) we have a wage stagnation or even slightly decreasing in some years.

Figure 4: Average wage in CR in different prices



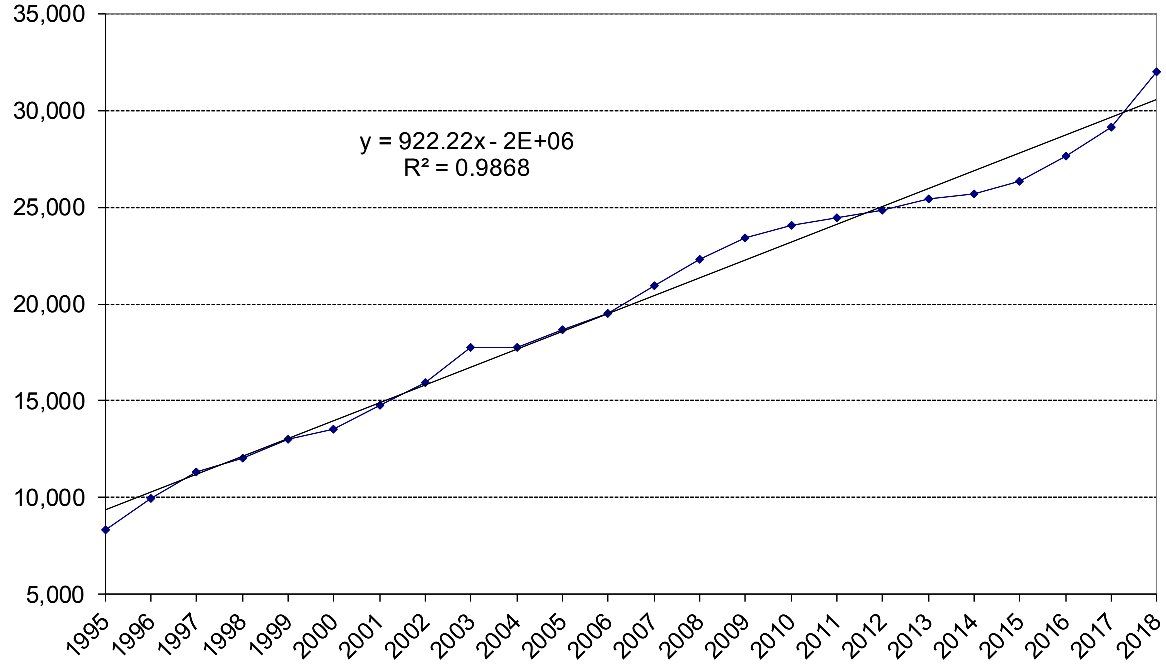
Source: Own graph

4.2 Trend analysis of wages

Let us have a look at the trend analysis of wages. We will best evaluate the trend of wage development graphically in Figure 5.

The trend is clearly linear. Model quality is very good. R-square value is 0.9868 (very close to 1) and we would certainly not find a better trend curve. In the early years, growth was strictly linear and the observed values were close to the theoretical curve. Approximately from the year 2007, we see little fluctuation around the trend. In the last year, wages have been rising steeply, which is related to increased demand for labor.

Figure 5: Trend of wages



Source: Own graph

4.3 Other quantile characteristics of wages

Other characteristics, based on quantiles, can also be used. First, let's mention *the trimean*, which is another characteristic of location calculated as a weighted average of quartiles:

$$\bar{x}_{trimean} = \frac{x_{0,25} + 2x_{0,5} + x_{0,75}}{4}, \tag{7}$$

where the median has a double weight against quartiles as we can see in Fig. 2. The values of this characteristic do not differ very much from the median. The difference is really very tiny and thus there is no reason to show this characteristic as another measure of wage location.

The difference between average, median and trimean we can see in Figure 5. We can observe that, as expected, the average of wages is the largest. Both the median and the trimean are significantly smaller, the latter somewhat larger than the former. The difference between each pair of characteristics is growing with time.

Other characteristics include variability measures, especially *the interquartile range:*

$$R = x_{0.75} - x_{0.25} \,. \tag{8}$$

This value is already in column 2 of Tab. 2. Another used characteristic is *the quartile deviation*:

$$R_2 = \frac{x_{0.75} - x_{0.25}}{2}. \tag{9}$$

We will use *the quantile skewness coefficient* that measures the quantile skewness:

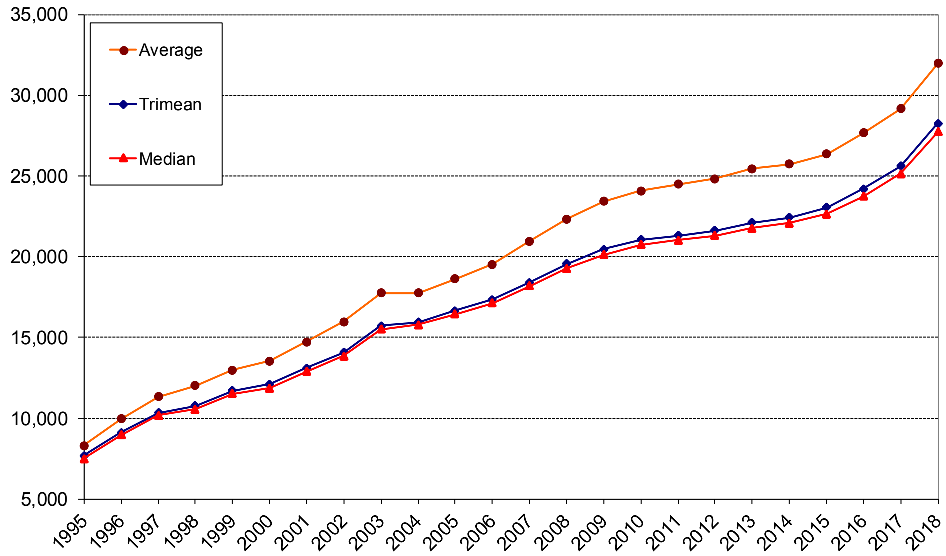
$$S_p = \frac{(x_{1-p} - x_{0.50}) - (x_{0.50} - x_p)}{x_{1-p} - x_p}. \tag{10}$$

where *0<p<1* and *p* is usually 0.1 or 0.25.

We will also calculate the quantile measure of kurtosis, the so-called *quantile kurtosis coefficient:*

$$K_p = \frac{x_{max} - x_{min}}{x_{1-p} - x_p}. \tag{11}$$

Figure 2: Average, median and trimean



Source: Own graph

Table 6: Quantile measures of variability, skewness, and kurtosis

Year	R_2	$S_{0.1}$	$S_{0.25}$	$K_{0.1}$	$K_{0.25}$
1995	1,864	0.295	0.176	20.806	41.496
1996	2,229	0.273	0.144	16.994	34.703
1997	2,587	0.246	0.126	14.600	29.906
1998	2,843	0.264	0.139	13.309	27.204
1999	3,026	0.268	0.125	12.273	25.563
2000	3,246	0.275	0.143	11.499	23.826
2001	3,462	0.284	0.125	10.639	22.345
2002	3,747	0.283	0.121	9.778	20.644
2003	4,121	0.255	0.104	9.032	18.772
2004	4,047	0.221	0.082	9.123	19.112
2005	4,486	0.232	0.102	8.415	17.241
2006	4,655	0.222	0.085	8.092	16.617
2007	4,971	0.252	0.090	7.411	15.559
2008	5,255	0.262	0.109	6.954	14.719
2009	5,451	0.277	0.120	6.608	14.190
2010	5,615	0.279	0.114	6.430	13.775
2011	5,615	0.279	0.105	6.320	13.777
2012	5,651	0.277	0.108	6.170	13.688
2013	5,863	0.285	0.116	5.909	13.192
2014	5,987	0.286	0.122	5.813	12.921
2015	6,138	0.288	0.125	5.691	12.602
2016	6,286	0.301	0.146	5.509	12.305
2017	6,532	0.308	0.144	5.271	11.843
2018	7,082	0.307	0.149	4.957	10.922

Source: Own calculations

We chose *p* to be 0.1 and 0.25, i.e. decile-based and quartile-based measures.

Based on Table 3 we can conclude the following:

- Variability measured based on the variation deviation keeps growing over time, the results are in compliance with classic moment measures (the standard deviation). This quantile measure thus only confirms what we already know from classic measures.
- Skewness is similar at the beginning and end of the analyzed period and the smallest in the middle of the analyzed period (around the year 2006). This does not correspond much to the empirical division of frequency of wages (Marek, 2010) and contradicts the classic moment skewness measures. Therefore, we should be somewhat cautious when using quantile skewness measures.
- Kurtosis keeps diminishing during the entire period of 24 years. This is in line with the empirical division of frequency of wages as well as with the classic moment kurtosis measures. Therefore, the kurtosis measure very well describes the behavior of wage division and can be recommended as a suitable measure.

4.5 High wages

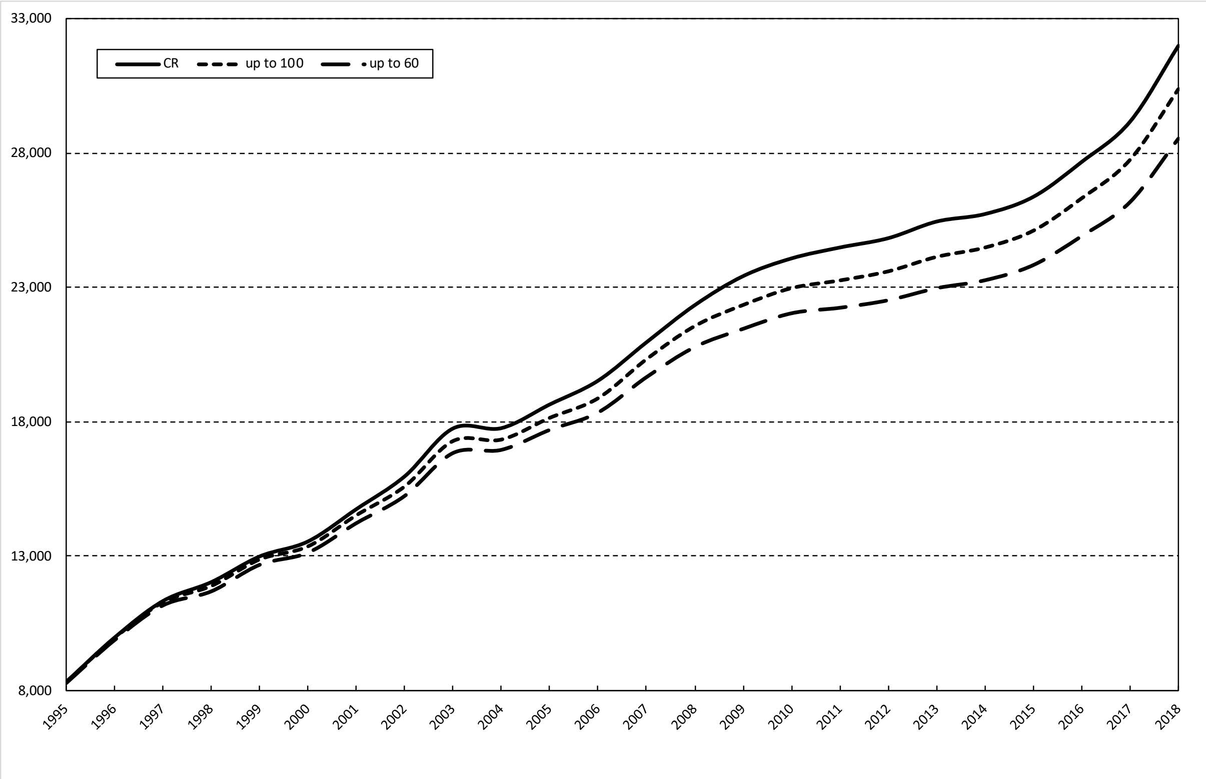
Under the name of *high wages,* we will understand wages above 100,000 CZK (approx. above 4000 EUR). Let's look again at Figure 1. The right-hand side, showing high wages, is of particular interest. The proportion of high wages was negligible in the early years, but this proportion has been growing steadily. There was only 0.0068% (practically zero) of wages above 100,000 CZK ("high wages" below) in 1995, but this proportion grew to 1.01284% in 2018. It is also interesting that there are very few wages between 60,000 and 100,000 CZK.

Table 4: Average wages after exclusion of high wages

Year	CR	up to 100	diff	relative frequencies	up to 60	diff	relative frequencies
1995	8,311	8,301	10	0.007%	8,273	39	0.177%
1996	9,962	9,932	31	0.026%	9,875	87	0.114%
1997	11,322	11,266	55	0.045%	11,167	155	0.202%
1998	12,026	11,874	152	0.121%	11,691	335	0.404%
1999	12,982	12,862	121	0.090%	12,679	303	0.385%
2000	13,541	13,347	193	0.132%	13,132	408	0.482%
2001	14,743	14,507	236	0.168%	14,230	512	0.626%
2002	15,964	15,581	383	0.257%	15,246	718	0.824%
2003	17,748	17,271	477	0.326%	16,843	904	1.070%
2004	17,759	17,325	434	0.288%	16,961	797	0.922%
2005	18,640	18,134	506	0.337%	17,705	935	1.096%
2006	19,526	18,864	662	0.420%	18,353	1,173	1.328%
2007	20,953	20,316	637	0.532%	19,668	1,285	1.711%
2008	22,338	21,552	786	0.648%	20,790	1,548	2.061%
2009	23,418	22,338	1,080	0.717%	21,469	1,949	2.348%
2010	24,077	22,964	1,113	0.753%	22,047	2,030	2.492%
2011	24,484	23,251	1,232	0.807%	22,252	2,231	2.711%
2012	24,829	23,589	1,240	0.850%	22,532	2,297	2.875%
2013	25,448	24,129	1,319	0.916%	22,981	2,467	3.138%
2014	25,728	24,475	1,253	0.909%	23,277	2,452	3.238%
2015	26,369	25,104	1,265	0.929%	23,842	2,527	3.414%
2016	27,668	26,310	1,358	1.013%	24,921	2,746	3.805%
2017	29,166	27,743	1,423	1.089%	26,188	2,978	4.305%
2018	31,992	30,375	1,617	1.284%	28,552	3,440	5.254%

Source: Own calculations

Figure 3: Comparison of average wages

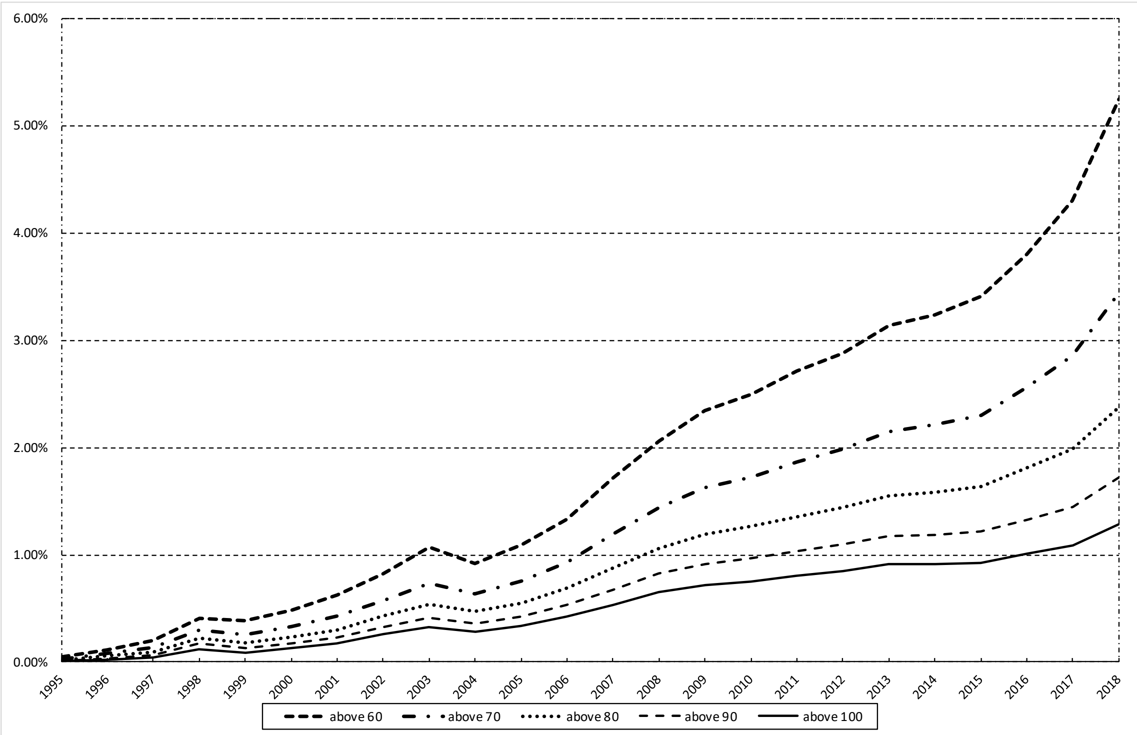


Source: Own graph

The "CR" column shows the officially published average wages; "up to 100" the average wages after the exclusion of wages above 100,000 CZK. The "diff" column is the difference between "CR" and "up to 100", that is, the difference of the average wages over the entire Czech Republic from the average after the exclusion of wages above 100,000 CZK. The meanings of the other columns are similar. If we exclude wages above 100,000 CZK, the average wage in 1995 goes down by a mere 10 CZK, while in 2018 this difference amounts to 1,617 CZK. A similar effect can be observed for other exclusions. From Figure 8 we might think that there are nearly no wages between 60,000 CZK and 100,000 CZK. However, after exclusion of wages above 60,000 CZK, the considered difference in 2018 is increased to 3,440 CZK, which is a substantial amount.

Figure 8 illustrates the growing gap between the average wages and the average wages after the exclusion of the high wages (above 100,000 CZK and above 60,000 CZK). For the sake of clarity, not all columns of Table 7 are shown in this Figure. Most extreme wages are reached in Prague. Many companies of the big (often foreign) companies run their business from there and the wages are mostly comparable to the wages in other countries. The wages of executive officers are usually much higher than the wages of other employees. In other regions, the effect is not so noticeable, as there are much fewer high wages - see (Marek, 2016). There are multiple factors influencing the value of high wages. A major factor is a sex - which will be demonstrated in the second part of the article. Other factors such as profession and education can be observed - i.e. the combination of tertiary education and IT profession implies higher than average wages. This is further discussed in (Marek, Doucek, 2016).

Figure 4: Proportion of high wages in the total amount

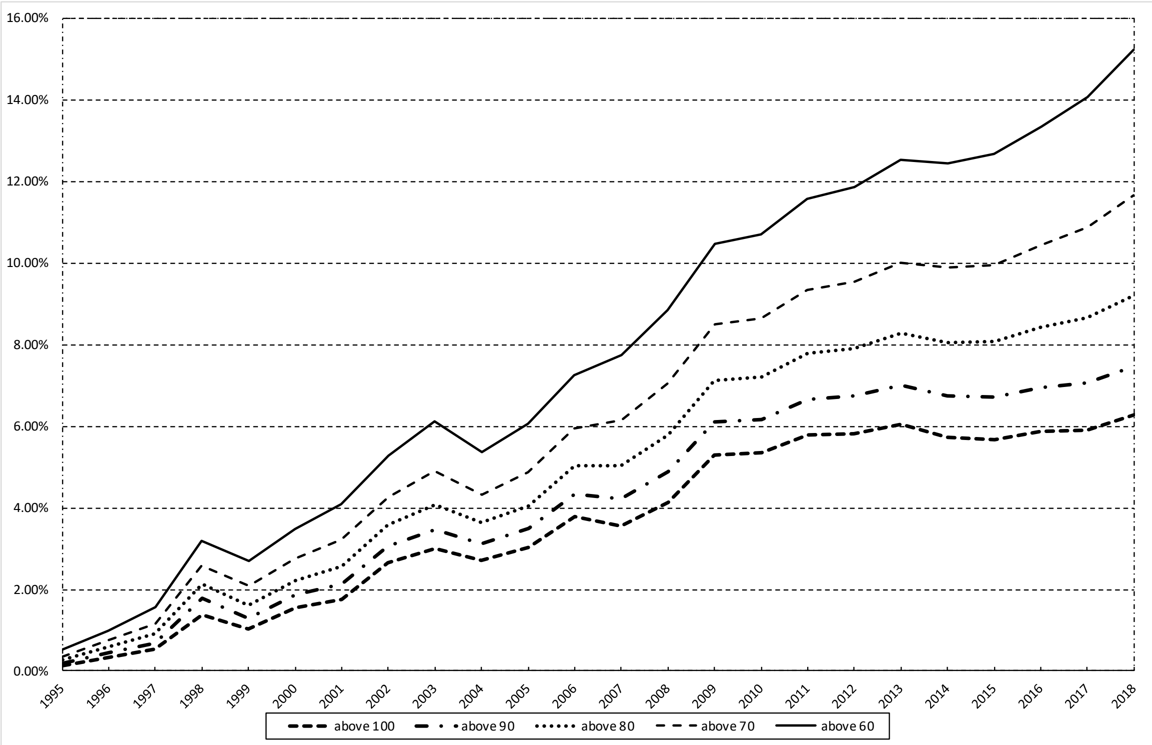


Source: Own graph

Figure 9 illustrates the growing proportion of high wages in the total amount. Similar to Figure 8 (columns "relative frequencies"), the growing gap between the wage categories is clearly visible as well.

We are also interested in proportions of High Wages and in the total amount of wages. Figure 10 clearly indicates that the effect of the wages between 60,000 CZK and 100,000 CZK is very small regarding the quantity, but the proportions regarding the total amount are much greater. Figure 10 illustrates the growing proportion of the high wages in the total amount. Similar to Figure 9, the growing gap between the wage categories is clearly visible as well. Multiple processes can be observed. The number of high wages increases with time. The share of high wages in the total volume of wages increases even faster than the number of high wages. It can be assumed that this trend will continue in the future. We are using data from 2016 to 2018 - there has been a significant economy growth in these years, which still continues. This growth causes a high demand for workers, which in turn causes the wages to grow. This will cause either the growth of the average wage and the number and importance of high wages. It will be worth making a similar analysis in 2-3 years and compare the results with the current ones. The problem of high wages is analyzed in detail in (Marek, 2019).

Figure 5: Proportion of high wages in the total amount



Source: Own graph

4.7 Average as Percentile

It is usually observed that two-thirds of employees do not achieve the average wage. This would put the average around a 67% percentile. We will verify on real data whether it is really true. The high wages obviously do affect the average – a natural question arises what level of percentile will correspond to the average after the exclusion of the high wages. Logically, the average should approach the median. Table 9 shows the results of the calculations in this direction. The "CR" column shows the official average wages, "up to 100" the average wages after exclusion of wages above 100,000 CZK, and "up to 60" after exclusion of wages above 60,000 CZK. The "percentile" column shows to what level percentile the respective average value corresponds. In the beginning years, the average is about a 63% percentile in all instances; the percentile value is then gradually growing. In the most recent years, the generally accepted assertion that the average wages are at the 67% level of percentile is confirmed for the "CR" column.

The situation is significantly changed after the exclusion of wages above 100,000 CZK. The percentile level of the average value remains between 61% and 63% for the entire period of observations. Let us now exclude the wages above 60,000 CZK. The average value gets down to 58% and is much closer to the median value. In other words, the percentile characteristics also confirm the assertions formulated above.

Based on the data, one can see that the average wage is the 67% percentile and thus the claim, that 2/3 of wages are lower than the average, holds. It will stay the same for the future. The wage distribution is positively skewed (due to high wages, which are outliers) and thus the mean will always be higher than the median. If the trend of wage distributions stays the same (which it did for the past 24 years), the rank of the mean as a percentile will further increase. It can be assumed that the mean will be a 68-69% percentile. However, growth will be rather slow. A rapid change is unlikely - as can be seen in Table 9, the rank of the mean as a percentile is very stable. It can be seen that for the past 5 years, the growth stagnates and the mean stays around the value of the 65-67% percentile.

Table 5: Average as percentile

year	CR	percentile	up to 100	percentile	up to 60	percentile
1995	8,311	0.632	8,301	0.632	8,273	0.632
1996	9,962	0.619	9,932	0.619	9,875	0.619
1997	11,322	0.631	11,266	0.631	11,167	0.632
1998	12,026	0.665	11,874	0.628	11,691	0.630
1999	12,982	0.626	12,862	0.627	12,679	0.629
2000	13,541	0.663	13,347	0.630	13,132	0.632
2001	14,743	0.652	14,507	0.653	14,230	0.656
2002	15,964	0.644	15,581	0.646	15,246	0.618
2003	17,748	0.652	17,271	0.627	16,843	0.602
2004	17,759	0.641	17,325	0.613	16,961	0.585
2005	18,640	0.644	18,134	0.620	17,705	0.597
2006	19,526	0.654	18,864	0.605	18,353	0.584
2007	20,953	0.635	20,316	0.614	19,668	0.596
2008	22,338	0.645	21,552	0.627	20,790	0.588
2009	23,418	0.655	22,338	0.616	21,469	0.580
2010	24,077	0.669	22,964	0.611	22,047	0.598
2011	24,484	0.659	23,251	0.623	22,252	0.588
2012	24,829	0.667	23,589	0.633	22,532	0.600
2013	25,448	0.664	24,129	0.632	22,981	0.578
2014	25,728	0.668	24,475	0.617	23,277	0.587
2015	26,369	0.660	25,104	0.629	23,842	0.580
2016	27,668	0.668	26,310	0.620	24,921	0.575
2017	29,166	0.667	27,743	0.622	26,188	0.581
2018	31,992	0.650	30,375	0.608	28,552	0.578

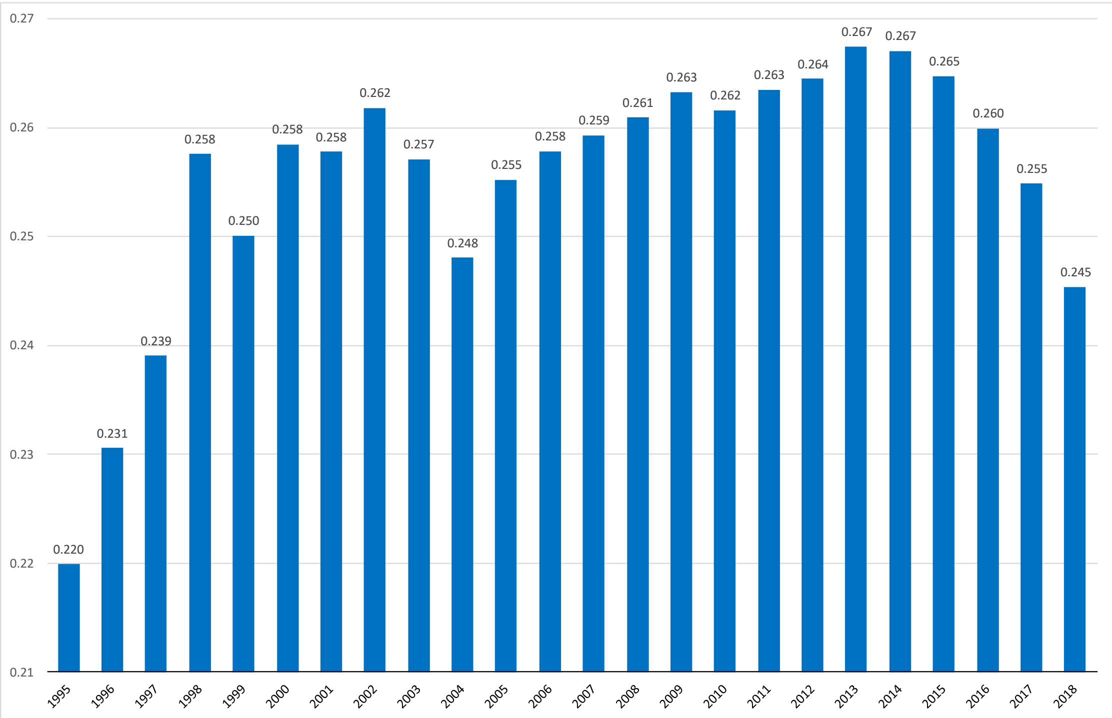
Source: Own calculations

4.8 Gini index

After the change of the regime in 1989, Czechoslovakia was a very egalitarian country. All wages were small and nearly the same. This situation has been gradually changing, and the wage variability began to grow. The wage inequality has thus been growing. We will make use of the so-called Gini index for measuring this inequality – this index was designed by Italian statistician Corado Gini (Gini, 1912). This index compares the actual and ideal Lorenz curves by calculating the ratio of areas below them (Marek, 2010). We have calculated the values of the Gini index for the CR data – the results of our calculations are shown in Figure 11.

The value of the index was rather variable in the beginning years. In CR, its value in long-term level has been fluctuating around 0.26. Contrary to all expectations, the value of the Gini index has been decreasing in the three most recent years.

Figure 6: Gini index



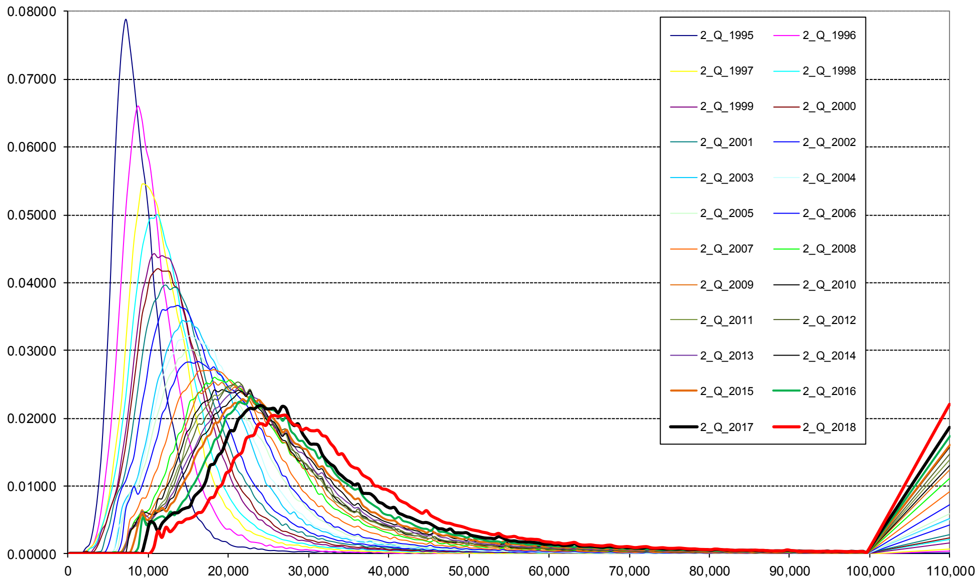
Source: Own graph

4.9 Wages by Gender

A number of factors influence the economic activities and the wages – cf. (Smyk, Tyrowicz, Liberda, 2014), (Guo, Yu, 2017), and (Dunsch, 2017). We will focus on gender, having a look at the evolution of the average wage values for men and women and possible gender differences.

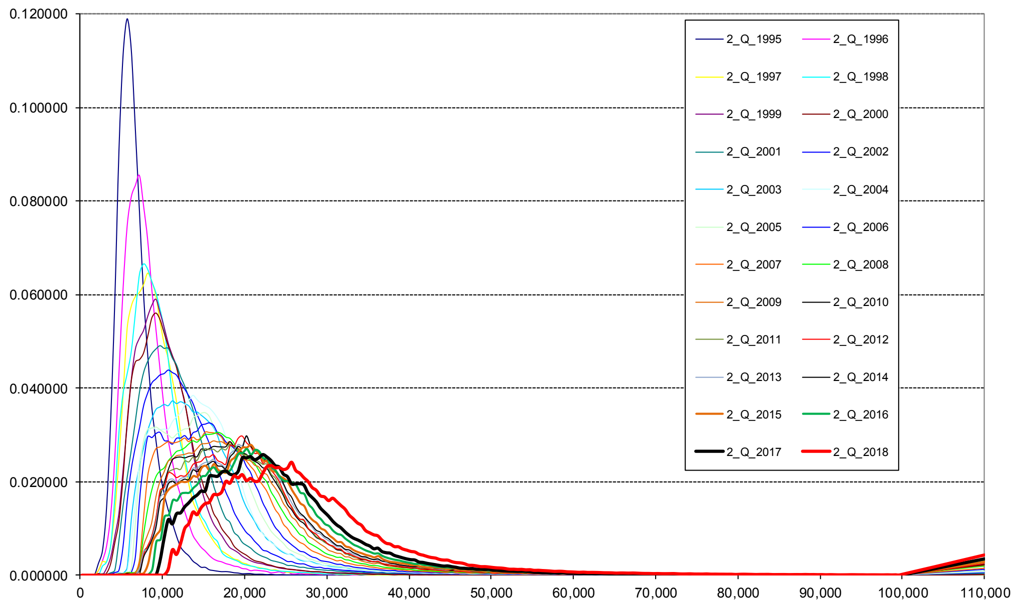
Let us first look at the empirical distribution of wages (Figures 12 and 13). These graphs are not deliberately presented on the same scale but the differences between men and women are visible at the first look. The most interesting features are both ends of the distributions. For both men and women, the data has a higher kurtosis in the beginning. The tails of the distributions are also different. There are more high wages (above 100.000 CZK) for men. The average values of wages are very similar for both sex, but the inner structures of the wage distributions are different. This difference cannot be expressed by the average; the empirical distribution of wage frequencies must be studied to reveal it.

Figure 7: Wage distribution – men



Source: Own graph

Figure 8: Wage distribution – women



Source: Own graph

The relevant values are shown in Table 11. The data indicate that the initial situation in wages was not about the same. It is clear that gender has a significant effect on average wages. The men's wages are significantly higher than women's. Women's average wage level in 2018 was at a mere 76,2% of men's. The number of high wages above 100.000 CZK is much higher for men as well. This fact is clearly indicated in Figures 12 through 13. The gap between both genders is evident from Table 11 and Figure 13 and is greater and greater in time.

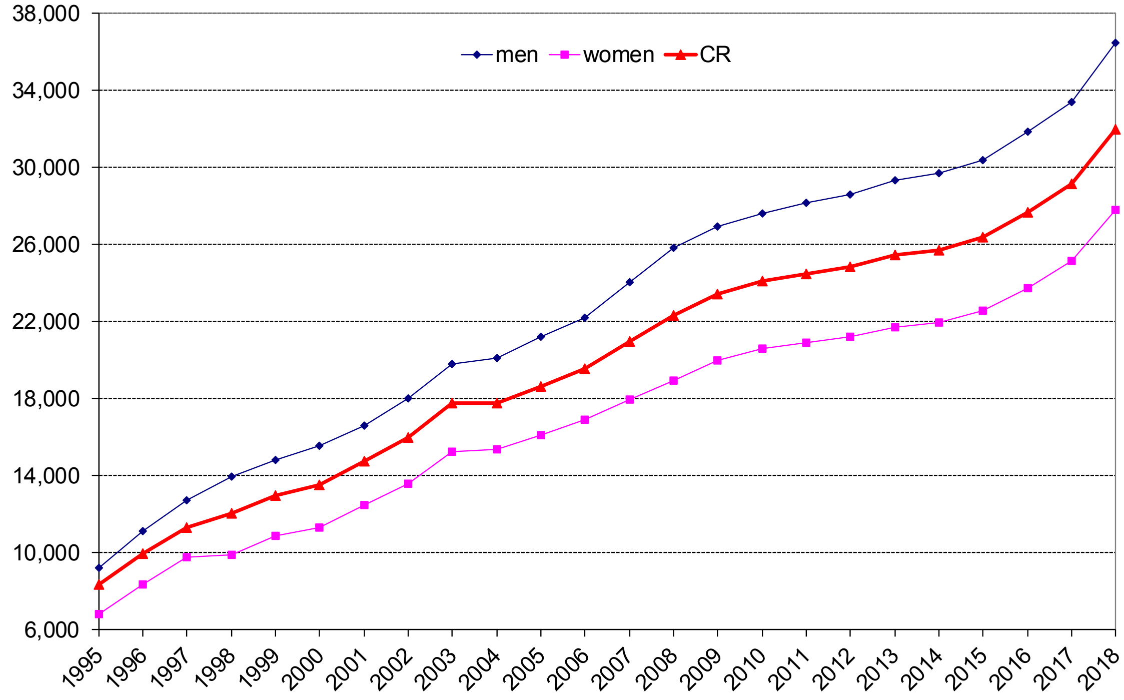
Table 6: Wages by gender

year	CR Men	CR Women	Difference
1995	9,221	6,794	2,427
1996	11,100	8,363	2,736
1997	12,737	9,740	2,997
1998	13,914	9,872	4,042
1999	14,835	10,878	3,957
2000	15,537	11,281	4,256
2001	16,580	12,435	4,144
2002	17,987	13,565	4,422
2003	19,784	15,217	4,567
2004	20,109	15,380	4,729
2005	21,188	16,076	5,111
2006	22,203	16,882	5,321
2007	24,026	17,916	6,110
2008	25,821	18,912	6,909
2009	26,929	19,957	6,972

year	CR Men	CR Women	Difference
2010	27,644	20,585	7,059
2011	28,196	20,903	7,293
2012	28,617	21,189	7,428
2013	29,360	21,694	7,666
2014	29,697	21,957	7,740
2015	30,376	22,569	7,806
2016	31,856	23,724	8,132
2017	33,424	25,171	8,253
2018	36,495	27,810	8,686

Source: Trexima

Figure 9: Average wage by gender

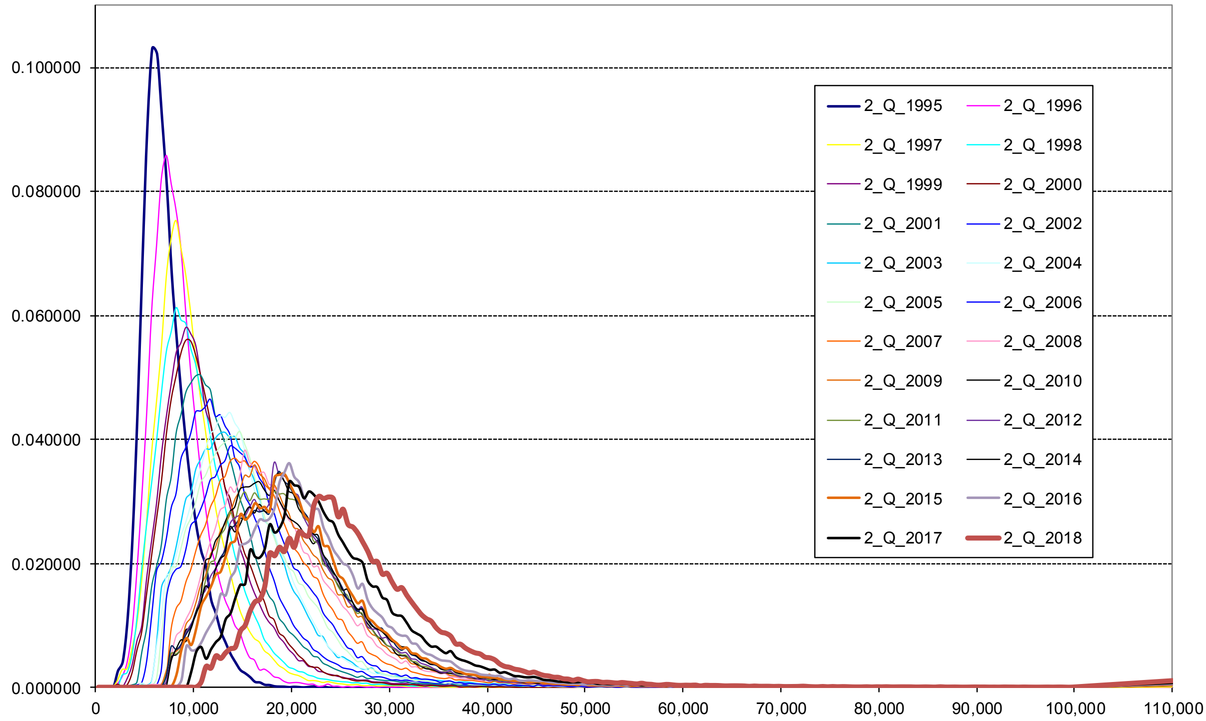


Source: Own graph

4.10 Wages by Age

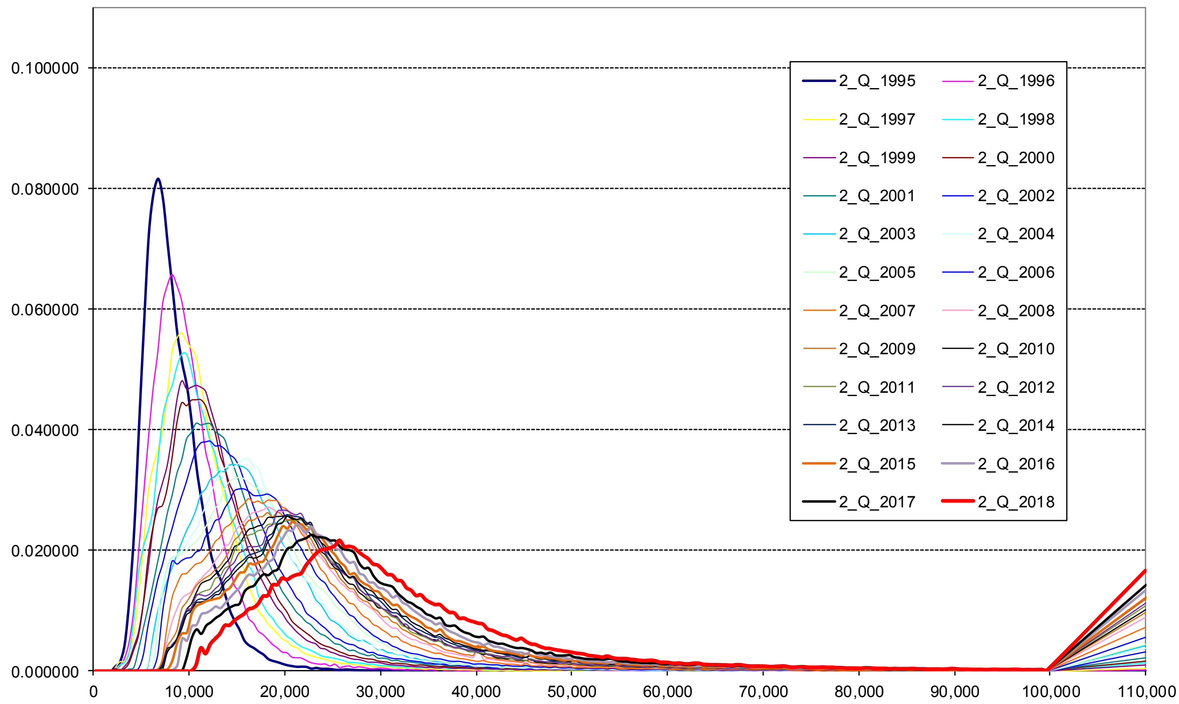
Let us study the effects of age on the average values of wages. We have at our disposal data for three age groups: up to 30, 30-50, and 50 plus years of age. We will again consider the average values of wages. We start with an empirical distribution of wages.

Figure 10: Average wages up to 30



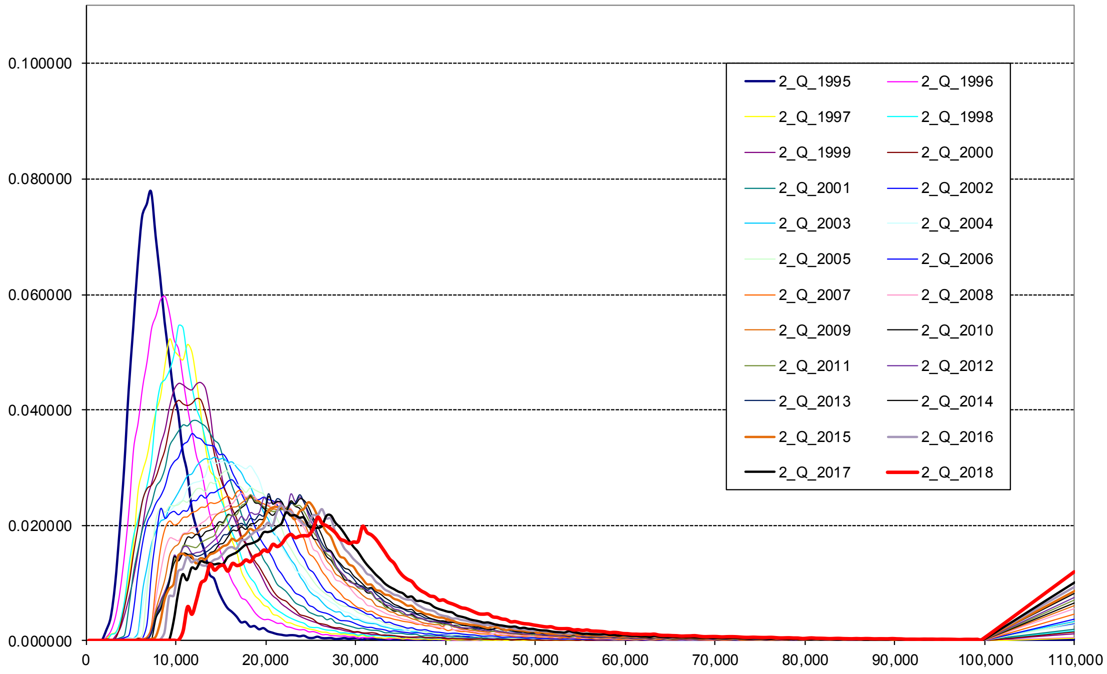
Source: Own graph

Figure 11: Average wages 30-50



Source: Own graph

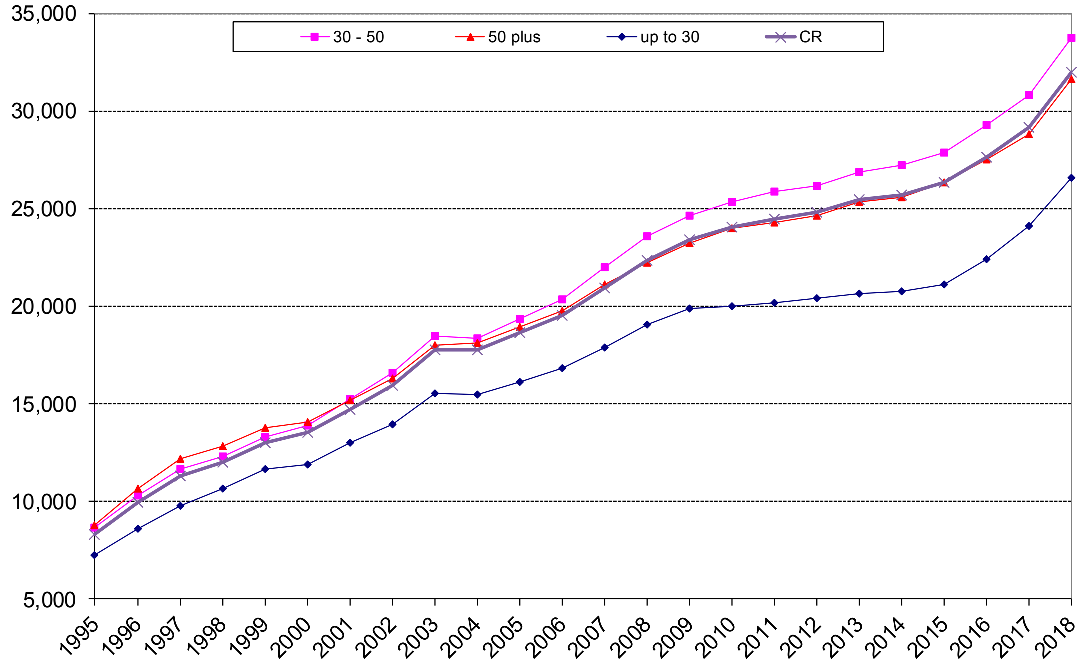
Figure 12: Average wages 50 plus



Source: Own graph

It is clear from the empirical distribution of wages that the situation is different for all three age groups. The most different is the group up to 30 years. The development of average wages will be better seen in Figure 19.

Figure 13: Average wages by age



Source: Own graph

The age group of up to 30 gets the smallest wages, followed by 50 plus; the highest wages are paid to the 30-50 age group. The difference between 30-50 and 50 plus is not so great. The 50 plus age group has an average wage almost identical to the national average. The wages in CR were growing faster in the beginning but were also faster to go down in the time of the economic crisis (Balcar, Gottwald, 2016).

The curve for the age group up to 50 years almost coincides with the average wage curve for the whole Czech Republic. The differences are minimal and almost indistinguishable on the graph. The best comparison is based on the age group 30-50, which has the highest wages. The group up to 30 years of age has, as expected, the lowest wages for the whole period of monitoring.

4.11 Wages by Education

Now we are working with data for 2000-2018. Our data is divided according to the education level – Basic, Secondary, Bachelor, Master, and Ph.D. There are 5 datasets at our disposal, and we can compare them with each other. Each dataset is a time series from the years 2000 through 2018, i.e., it contains 19 items of data.

Table 7: Simple Measures of Dynamics

Education Level	$\overline{\Delta}_t$	$\overline{k}_t$
Basic	697	1.0503
Secondary	1,001	1.0499
Bachelor	1,172	1.0505
Master	1,423	1.0460
PhD	1,678	1.0513

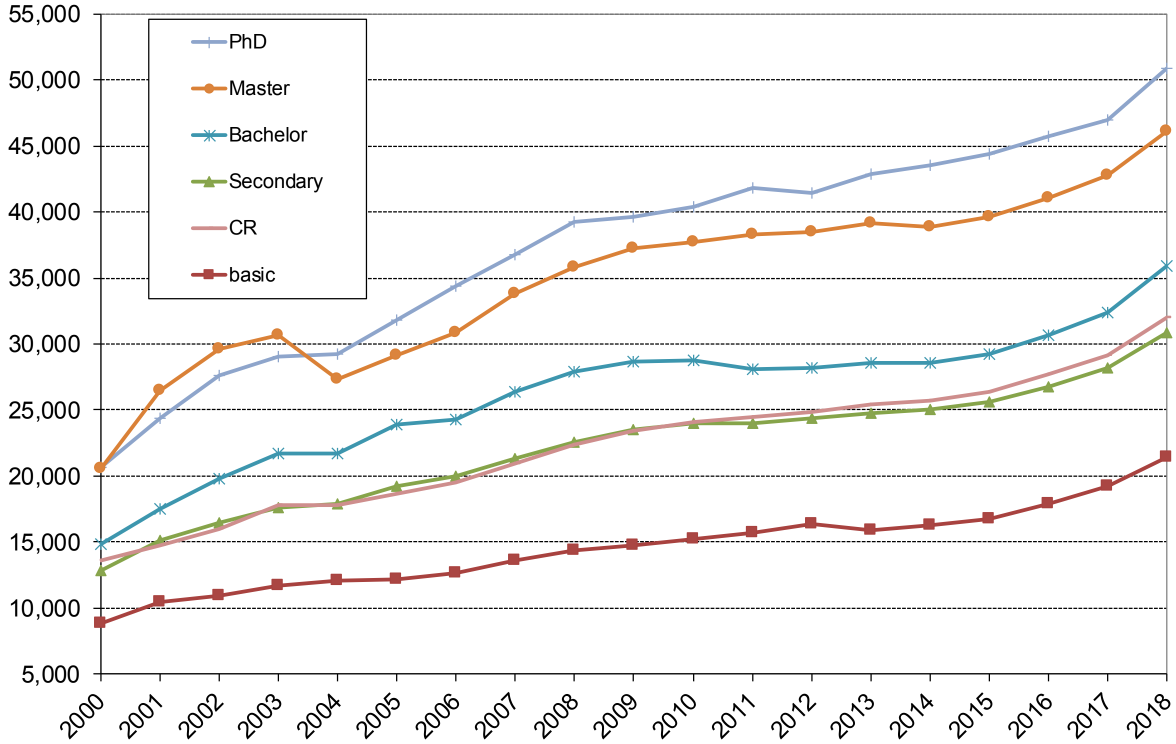
Source: own calculations

Several observations are implied by this Table 14:

- the smallest value of the average absolute increment occurs in the Basic education level;
- it is true that the absolute growth of the wages is larger for a higher education level;
- by far the largest average absolute increment occurs in the Ph.D. category;

Let us have a look at the time evolution of average wages in the Czech Republics in the most recent 18 years with regard to the achieved levels of education.

Figure 14: Wages by education



Source: own graph

The explanations in the graphs go top-down. The influence of the education level is clear at the first look. The ordering of the wage values follows the education levels, namely: Basic < Secondary < Bachelor < Master < Ph.D. We can therefore state:

- education level affects the average wage values;
- in general, the higher the education level, the higher the average wage values;
- the highest wage values occur in the Ph.D. category;
- it is very interesting that the wage level for secondary education is almost identical to the national average wage.

A good view of the statistical properties of all the monitored categories will enable the Boxplot shown in Figure 22.

5 Models of wage distributions

Various probabilistic distributions can be used to model empirical wage distribution. If the wage distribution is more or less "smooth", it can be adequately modeled with the aid of a single distribution. But as the wage variability grows over the years and empirical density curves became less smooth, mixture models have the potential to provide better models of wage distributions in the future. Concerning single distributions, we used Log-normal, Log-normal (3p), Johnson SB, Log-Logistic, Log-Logistic (3p).

The log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. The parameters of the distribution are:

s - continuous parameter (*s* > 0),

m - continuous parameter,

g - continuous location parameter (*g* = 0 yields the two-parameter Lognormal distribution) and the domain is $\gamma < x < +\infty$.

The three-parameter Log-normal distribution has the probability density function

$$f(x) = \frac{\exp\left(-\frac{1}{2}\left(\frac{\ln(x-\gamma)-\mu}{\sigma}\right)^2\right)}{(x-\gamma)\sigma\sqrt{2\pi}} \tag{12}$$

and the cumulative distribution function

$$F(x) = \Phi\left(\frac{\ln(x-\gamma)-\mu}{\sigma}\right) \tag{13}$$

The two-parameter Log-normal distribution has a probability density function

$$f(x) = \frac{\exp\left(-\frac{1}{2}\left(\frac{\ln(x-\mu)}{\sigma}\right)^2\right)}{x\sigma\sqrt{2\pi}} \tag{14}$$

and cumulative distribution function

$$F(x) = \Phi\left(\frac{\ln(x-\mu)}{\sigma}\right) \tag{15}$$

where Φ is the Laplace Integral.

Johnson distributions (Johnson, 1949) are based on a transformation of the standard normal variable. Given a continuous random variable X whose distribution is unknown and is to be approximated, Johnson proposed three normalizing transformations having the general form:

$$Z = \gamma + \delta\, f\left(\frac{X-\xi}{\lambda}\right), \tag{16}$$

where f (.) denotes the transformation function, Z is a standard normal random variable, γ and δ are shape parameters (δ > 0), λ is a scale parameter (λ > 0) and ξ is a location parameter. We will consider the Johnson SB distribution where

$$Z = \gamma + \delta \ln\left(\frac{X-\xi}{\xi+\lambda-X}\right). \tag{17}$$

The domain of this distribution is 0 < y < 1, the density function is

$$f(y) = \frac{\delta}{\sqrt{2\pi}} \frac{1-y}{y} \exp\left(-\frac{1}{2}\left(\gamma + \delta \ln\left(\frac{y}{1-y}\right)\right)^2\right), \tag{18}$$

and the cumulative distribution function is

$$F(y) = \Phi\left(\gamma + \delta\, ln\left(\frac{y}{1-y}\right)\right), \tag{19}$$

where $y = \frac{x-\xi}{\lambda}$, and Φ is the Laplace integral.

Log-logistic distribution is the probability distribution of a random variable whose logarithm has a logistic distribution. The parameters of the distribution are

a - continuous shape parameter (*a* > 0),

b - continuous scale parameter (*b* > 0),

g - continuous location parameter (*g* = 0 yields the two-parameter Log-Logistic distribution) and the domain $\gamma \le x < +\infty$.

The three-parameter Log-logistic distribution has the probability density function

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-2} \tag{20}$$

and the cumulative distribution function

$$F(x) = \left(1+\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right)^{-1}. \tag{21}$$

The two-parameter log-logistic distribution has the probability density function

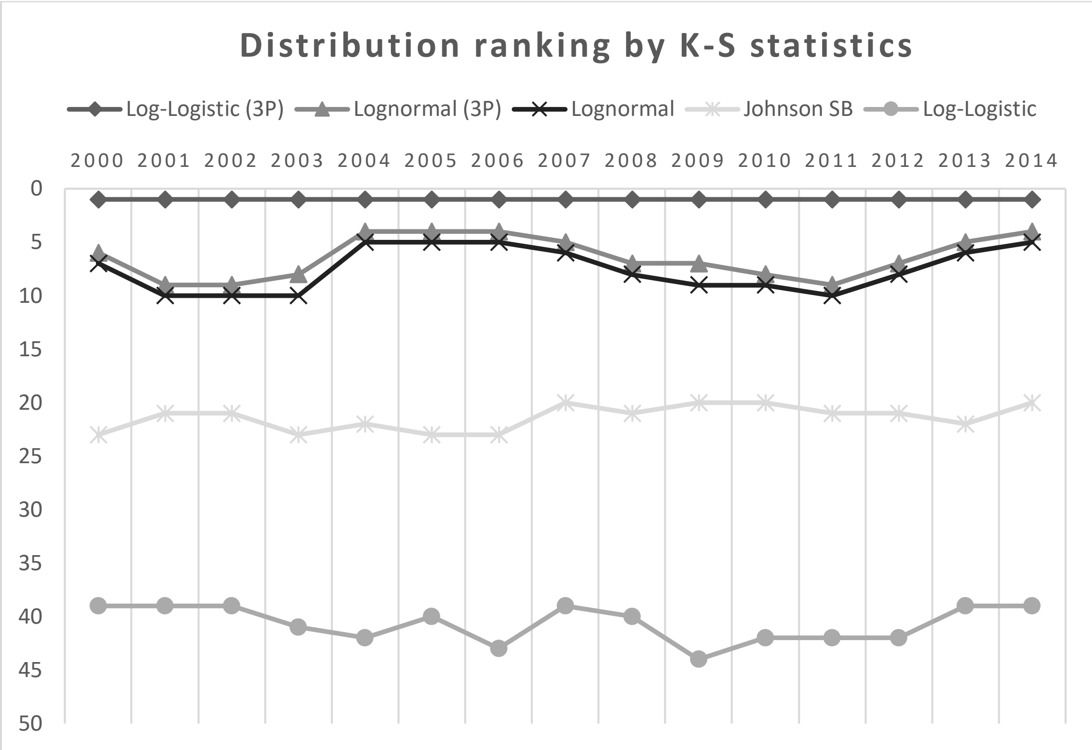
$$f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} \tag{22}$$

and the cumulative distribution function

$$F(x)=\left(1+\left(\frac{\beta}{x}\right)^{\alpha}\right)^{-1}.$$

The selection of the distributions was based not only on the fact that these distributions are widely used to model wage distributions but also on our modeling experiments of wage distributions for the period 2000-2014. Fig. 31 summarizes the results of these experiments. Here each curve shows the rank for the used distributions assigned according to the value of the Kolmogorov-Smirnov statistics to more than 50 probabilistic distributions available in the EasyFit system. The average rank for three-parameter Log-Logistic distribution was 1.0 (this distribution was always the best one), the average rank for three-parameter Log-Normal distribution was 6.4, the average rank for two-parameter Log-Normal distribution was 7.5, the average rank for Johnson(SB) distribution was 21.4 and the average rank for two-parameter Log-Logistic distribution was 40.7. Among other distributions reported in the literature as suitable to model wage distributions, the Dagum distribution (Dagum, 2008), used e.g. by Matějka and Duspivová (2013) had the average rank 53.2 and therefore was not included in the prediction experiments.

Figure 31: Ranking of distributions based on Kolmogorov-Smirnov statistics 2000 - 2014



Source: Own graph

Concerning the mixture model, we used Normal Mixture distributions with a different number of components. The probability density for a general model of a normal mixture can be written as

$$f(x) = \sum_{i=1}^{n} p_i g_i(x), \tag{24}$$

where $g_i\,(x)$ is the probability density of normal distribution

$$g_i(x) = \frac{1}{\lambda_i\sqrt{2\pi}} \exp\left(-\frac{(x-\theta_i)^2}{2\lambda_i^2}\right), \tag{25}$$

n is the number of components in the mixture and p is the vector of weights, for which

$$0 < p_i < 1,\ \forall i,\ \sum_{i=1}^{n} p_i = 1 \quad . \tag{26}$$

We create not only the models of wage distributions for historical data but also perform a set of prediction experiments, where the models have been used to predict the empirical wage distributions for the future years. We carried out three prediction experiments (Marek, Vrabec, Berka, 2018):

- wage data for the period 1995-2016 have been used to predict the parameters of the distributions for the year 2017 (Prediction1),
- wage data for the period 1995-2015 have been used to predict the parameters of the distributions for the years 2016 and 2017 (Prediction2),
- wage data for the period 1995-2014 have been used to predict the parameters of the distributions for the years 2015, 2016 and 2017 (Prediction3).

When working with a single distribution, we created one model for each prediction, when working with a mixture, we created a mixture model with two components reflecting gender (male, female). Distributions based on predicted parameters have been compared with the empirical wage distribution in the year 2017; we performed the Kolmogorov-Smirnov test testing the null hypothesis "H0: the data follow the specified distribution created using the predicted parameters" against the alternative hypothesis "H1: the data do not follow the specified distribution created using the predicted parameters". In all these predictions, the parameters were predicted using a linear trend.

Table 16 shows the quality of prediction for 2017 in terms of the Kolmogorov-Smirnov statistics and the rank of the model. A common expectation is that the more ahead a prediction is made, the less reliable it will be. So in our experiments, we expected that the Prediction1 experiment will give the best results and the Prediction3 experiment will give the worst results. But this expectation was not confirmed by the values of the Kolmogorov-Smirnov statistics. Fig. 3 illustrates the fit of the respective model for Prediction1 (i.e. model created from the years 1995-2016 predicts for the year 2017). We used the SAS system, JMP and EasyFit programs for the computations.

Table 16: Results of the Kolmogorov-Smirnov test

model	Prediction1		Prediction2		Prediction3	
	statistic	rank	statistic	rank	statistic	rank
Log-normal (3p)	0.03886	3	0.03809	3	0.03739	3
Log-normal	0.13290	5	0.15408	5	0.18248	5
Johnson SB	0.06210	4	0.06605	4	0.06982	4
Log-logistic (3p)	0.01900	1-2	0.02872	2	0.01830	1
Log-logistic	0.20191	6	0.21899	6	0.23095	6
Mixture (2comp)	0.01900	1-2	0.01961	1	0.02223	2

Source: Own calculations

Our prediction models were created using the most simple way, by a linear trend. More advanced methods like a nonlinear trend or Holt exponential smoothing can be considered as well (and this can be a possible direction of our future work) but even the linear trend gave the values of R2 varying from 0.9704 to 0.9937. When comparing the results of prediction experiments for any of the used model, we do not see any great difference in goodness of prediction for the year 2017 based on the data from the period 1995-2016 (Prediction1), based on the data from the period 1995-2015 (Prediction2) and based on the data from the period 1995-2014 (Prediction3). The reason could be the stable economic environment in the Czech Republic in the last years in which linear trend well fits the parameters of the wage distribution. When working with a normal mixture model, we considered only two components (males, females) because the categorization by gender has a high impact on wage distribution (see e.g. Bílková, 2012). But all the other natural components, that are presented in Section 4 can be considered as well. Some initial experiments in this direction are reported in (Marek, Vrabec, 2013). The above-mentioned categories can be used not only separately, but also simultaneously thus resulting in a mixture model with 2x3x5 components. Such a model will be of course computationally very complex and will require to process data on a very detailed level but has the potential to fit well the empirical wage distribution using an interpretable mixture model.

6 Summary

Our detailed analysis of empirical wage distributions in CZ for the period 1995-2018 gave the following main results:

- The average wage increases linearly over time if we consider it in current prices of a given year. When taking into account the inflation and prices in 2018, the increase slows down, the time series of average wages even slightly decreases in several periods.
- The wage distribution significantly changes all its characteristics over time – location, variability, skewness, and kurtosis.
- We can observe the highest increase of wages in the first two years and in the last year of observations.
- The average wage is significantly affected by high wages (wages above 100 000 CZK). When excluding high wages, the average wage drops by about 2000 CZK.
- A similar effect on average wage has the capitol Prague. Average wage outside Prague is about 2000 CZK smaller. The majority of high wages can be found in Prague.
- The average wage is affected by all the discussed factors – sex, age, education, region.

Concerning the future development of wages, we can expect that the increase will slow down in general, that high wages will increase and that the scissors of individual characteristics will widen.

Our modeling experiments show that among single probabilistic distributions the Log-logistic distribution with three parameters is the most suitable one to model empirical wage distribution. A normal mixture model with two components gives comparable results. Our prediction models were created using the most simple way, using a linear trend. More advanced methods like a nonlinear trend or Holt exponential smoothing can be considered as well. When comparing the results of prediction experiments for any of the used model, we do not see any great difference in goodness of prediction for the year 2017 based on the data from the period 1995-2016 (Prediction1), based on the data from the period 1995-2015 (Prediction2) and based on the data from the period 1995-2014 (Prediction3). The reason could be the stable economic environment in the Czech Republic in the last years in which linear trend well fits the parameters of the wage distribution.

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V

The impact of Digitalization on Defining and Measuring Some Basic Concepts in International Taxation

Monika Šestáková

1 Introduction

Present paper is from the field of a Philosophy of measuring some basic concepts in social sciences, particularly concepts in finance and accounting that are important for public policy and economic theory as well. Of course, the measuring and (re)defining the basic concepts in taxation and also in other problem areas that form a theoretical background for political decisions, is connected with *the role of a human factor* and its increasing importance under conditions of digitalization.

In the philosophy of measurement as a multidisciplinary field, the crucial problem is whether measurement outcomes in different disciplines "reflect the facts about the nature or about human tools and concepts" (CRASH, 2015). In economics, it often happens that numbers produced by economists do not adequately reflect the reality, Often, not sufficient attention is paid to the question what assumptions and limitations these data entail. According to the normative conception of good measurement in economics, the measured data must satisfy at least one of the following conditions: *(i) The number is based on the conceptually sound combination of quantities, (ii) The concept expressed by the number is numerically representable, (iii) The number is based on reliable and objective procedure. (Heilmann, 2015)*

The present paper focuses on the first condition and tries to show that traditionally provided data in the analyzed field are not "conceptually sound". To a high degree they are not "numerically representable". The third condition is a task for the future, it can represent the next step. Although current development in IT provides technical tools to perform these tasks, the institutional framework for such measurement must be created and based on a consensus among countries. *Human factor* is important in all these activities. Digitally skilled workers (in the form of human capital) not only help companies to improve their performance and better satisfy the current needs of the society. They are also important for government and other public institutions to materialize the digital transformation of the whole society and - in the field of taxation – to use up-to-date digital technology in monitoring the behavior of businesses, their tax compliance and suggest more effective forms of regulation. All the mentioned tasks, although performed locally, at the territory of individual countries, are to be planned and implemented taking into account the global context. International knowledge sharing and knowledge transfer at different levels is of key importance.

In author´s opinion, before applying exact and sophisticated methods to measure some phenomena, it is important to define the features of the object to be measured and redefine them if the traditional definition does not correspond to changes in reality.

Digitization of the economy leads to radical changes in implemented business models and their macroeconomic impact. During the last years there has been an intense debate between political institutions, international organizations, researches and technical experts on the impact of the digital business models, especially in the field of taxation. It is clear that some traditional theoretical and practical approaches in taxation do not correspond to a new reality.

Although very differing views have been presented in recent debates on the tax challenges of the digital economy, the consensus has been achieved in two important results (OECD 2017):

1) The so called digital economy can't be today separated (ring-fenced) from the traditional "physical" economy because new models digitize also the cycles of traditional products and services;

2) The different business models have different repercussions on value creation in different countries and change the taxing power of individual states. Of special importance is the fact that multinational enterprises may create value (and taxable income) in countries where they are not physically present.

Present paper will focus on direct taxes, especially corporate income taxation. Indirect taxes are not analyzed, although the digitalization process withdraws the exact dividing line between direct and indirect taxes.

Traditionally, the key concept defining the *nexus* between a host country (market country) where a business activity of a foreign investor (resident from a different country) is performed, and local authorities´ taxing power on the value created within their territory, has been the *concept of permanent establishment*.(Sestakova, 2018b). If such a permanent establishment does not exist, the resident country (of foreign investors) has the right to collect taxes from the business activity, even if the value has been created abroad.

Because of the key role of the permanent establishment concept in international taxation, the large part of the chapter is devoted to the redefining or recent modifications of this concept. However, there are also other aspects of the digitization impact in the field of business taxation.

One of them is an intensive *reliance of highly digitized businesses on intangible assets, including intellectual property.* As well known, measuring the intangible assets and intellectual capital belongs to the not appropriately solved problems in accounting and theory of taxation (Bernadič, 2016). We shall briefly discuss this problem in the further text.

Another widely discussed problem is the crucial *role of data in the new business model, user participation and their synergies with intellectual property. (*UiO 2017) Are data stemming from local users (suppliers, customers, etc.) a factor contributing to a taxable value creation?

The structure of the chapter is as follows. In the second- the largest part of the chapter - the critical approaches to the permanent establishment concept are surveyed and alternatives to the concept (such as virtual permanent establishment) are analyzed. The third part deals with some other taxation problems and approaches to define the value creation aspects of the new business model. .In the closing part of the chapter the author tries to show that the mentioned problems are topical also for Slovakia as a small, highly opened economy. Although the physical presence of foreign investors is still very important, the digital transformation significantly influences their business models and creates some challenges for a tax policy. *The impact of digitalization is especially important in the human capital trends,* where multinational companies employ and further train digitally skilled workers, improve their digital competences and by this way contribute (also by spill-over effects) to forming the labor force required for a digital transformation of the whole economy.

2 The "permanent establishment" concept – traditional definition, modifications, alternatives

Irrespective of the difficulty to find some stabile (relatively long.-term valid) norms in international taxation, certain basic, generally accepted features of key concepts can be found (OECD Convention 2014). This is true also for the permanent establishment (PE) concept.

Primarily, a PE requires *a fixed place* of business within the geographical boundaries of the country of sales. Fixed place may include administrative offices, factory, or workshop, but not necessarily sales offices or storage facilities. Under most double taxation treaties, the income of a firm is taxed in the country where it has permanent establishment only if it carries on there a business of a *continuing and lasting* kind. It is usually assumed that a foreign organization operates from a fixed place, an employee's job title or description indicates that he or she performs activities related to revenue generation or sales of the organization, and that employee operates in the host country for a prolonged period, sales are made to customers based in the host county and local contracts are negotiated by a locally-based employee or dependent agent.

Another problem is how to interpret time of "permanency" or how long the organization should perform business activities in the country and when (in the case of building a new enterprise) the taxing right of the host country authority can be applied. There are different views regarding this issue. Many treaties provide specific rules with respect to construction sites. Under those treaties, a building site or a construction or installation project constitutes a PE only if it lasts more than a specified length of time (e.g. 2 years).

According to the traditional definition, a PE need not be a legal entity, but it is treated as a functionally separate and independent entity, an object important for the fiscal policy.

Different alternatives to the nexus concept were suggested in the literature. Some of them just modify the traditional PE concepts. According to OECD (MTC 2014 article 5) even when there is no physical presence of the company in the market country, PE can exist if the agent "habitually exercises" a right to conclude contracts on behalf of the foreign company. It means that the foreign producer (supplier) can sell products and services in another country using intermediaries to solicit sales and persuade customers to enter into contractual relations with the foreign company. The further discussion focused on the interpretation of the word "habitually" and "independent agent". (Medus, 2016). This is especially important in the case of retail and consumer products oriented multinationals. According to some suggestions, there can be "a dependent agent PE" which exists, if there is a contract between the foreign company and local agent .However, it is difficult to achieve consensus in defining the legal form and time limits of such a contract.

Another modification of the nexus concept is connected with *Specific activity exemptions* (in the traditional PE definition) *and anti-fragmentation rules* (BEPS Action 7, OECD 2017). As mentioned above, the auxiliary activities were excluded from the PE definition. Nowadays these activities play a very important role in value creating process and can significantly contribute to profitability.

Currently, most R&C multinationals are involved in online sales, with some international sellers engaged solely in digital sales. Online sales usually require that an enterprise maintain a warehouse abroad (with an adequate number of employees) where goods owned by the enterprise are stored and delivered to local customers. According to new suggestions, these local facilities are to be treated as a PE for tax purposes.

Moreover, preparatory and auxiliary activities controlled by the same company can be located in different places within the same country and between countries. This *fragmentation* is a further problem for tax authorities. A multinational company can split the functions of cohesive business operations between different legal entities with physical presence in different states. Sometimes, these entities are not independent accounting units, value added by their activities is not clear and the tax liability is difficult to define.

Attempts of altering the original PE concept by different partial modifications are very frequent and contribute to making the concept more suitable for modern economic era. But usually they are dealing with separate, partial issues and do not provide a coherent alternative.

Perhaps, the concept of the Significant Economic Presence nexus (as also addressed in the ECOFIN conclusions of 5 December 2017) is the most complex and widely discussed alternative. This European Commission legislative proposal (EC 2017) suggests to take into account three groups of factors: revenue-based, user-based and digital.

Regarding the revenue-based and user-based factors, each state should define the threshold from which the remotely controlled transactions are ”significant”. Such threshold can be expressed as the minimum number of users of some platforms, value of cross-border e-sales, the proportion of total revenues obtained from the supply of digital services to customers in the host country, number of business contracts for the supply of digital services, etc.

Let´s mention at least some important thresholds suggested by the European Commission to consider a company as having significant digital presence in a Member state (EU 2018):

- It exceeds a threshold of €7 million in annual revenues from digital services in a Member State
- It has more than 100,000 users who access its digital services in a Member State in a taxable year;
- Over 3000 business contracts for digital services are created between the company and business users in a taxable year.

These criteria are regarded just as recommendations to member states and states as independent fiscal entities can define the threshold themselves. Finally the rules should be approved by national parliaments.

Regarding digital factors, proposed concept characterizes the *virtual permanent establishment* as the situation “when a non-resident taxpayer provides access to or offers a *digital platform* such as an electronic application, database, online market place, storage room, or offers search engine or advertising services on a website or in an electronic application" (EC 2018). Other potential digital factors are a local domain name or local payment options.

Once the significant economic presence is determined in different jurisdictions, it is necessary to determine (i) what creates value and (ii) where that value is created in order to allocate (taxable) profits to these jurisdictions.(Brauner & Pistone 2018)

The basic problem connected with the practical implementation of the "significant economic presence" concept (or virtual PE) is the necessity to achieve consensus between EU member state in defining the criteria and include them into bilateral tax treaties. Even if this barrier will be overcome, it will be difficult to implement the criteria in tax treaties with nonmember countries.

A short-term and unilateral response to tax challenges of digitized economy is the *equalization tax* – an additional tax on the turnover of digital economy businesses., Within the EU The French-led proposal is to apply in each EU Member State a 2%-6% levy on the turnover achieved by digital means in that State. This tax would target companies that do not have a taxable presence under current rules despite significant (virtual) interactions with clients and users.

This proposal does not solve theoretical problems of value creation under conditions of digitization, neither suggests criteria how to divide the revenues from the up-to-date value creating process between involved countries. It just aims at increasing taxes on 'all untaxed or insufficiently taxed income generated from all internet-based business activities, including business-to-business and business-to-consumer'. (Loyens/Loeff 2018).Similar tax exists in some non EU countries (e.g. India).

The equalization tax is a relatively easy to implement method, but it has important drawbacks. It actually taxes a turnover, not profits. It can lead to double taxation and incompatibility with international trade rules, administrative burden of implementing such a tax can be heavy, It is difficult to achieve an international consensus on the basic threshold for applying this additional tax. Anyway, it cannot be a long-term solution.

3 Valuating and measuring some other aspects of value creating process under conditions of digitalization

The role of intangible assets and intellectual capital

As widely known, the proportion of intangible assets in the total assets of digitalized companies is very high and intellectual capital is a crucial driver of their growth and increasing competitiveness. According to UNCTAD (UNCTAD. WIR 2016) the average market capitalization of digital megacorporations is almost three times higher than that of other top 100 MNEs. As estimated, such market capitalization can be largely attributed to highly valuable unreported (undisclosed) intangibles, such as brand, know-how and intellectual capital, that lead to a wide gap between market value and asset book value. According to UNCTAD, undisclosed intangibles form in the group of highly digitized companies about 91% of assets book value.

This high proportion of undisclosed assets is not only a result of manipulating with the reported tax basis. It is also a consequence of unsolved problem of intellectual capital definition and measuring both in theory and accounting practice. As B. Bernadič shows (Bernadič, 2016) there are different definitions of the intellectual capital in literature and no uniform definition has been suggested. Some authors even disagree with the opinion that a standardization of Intellectual capital valuating methods is necessary (Andriessen 2003 – according to Bernadič 2016) because to solve practical problems, diversity in solutions is important.

However, practically in all definitions, the concept is connected with knowledge, information and intellectual property that can be converted into profits. In the mentioned UNCTAD research, intellectual capital is actually understood as the difference between the market value of the corporation and its tangible assets. This approach can be more appropriate for large digital publicly traded corporations (basically US based) where the market value of the company can be objectively given. However, investor assigned market value is connected with consolidated financial statements, not with the value creation in the different regional units of the global corporation. In some countries, where capital markets do not work effectively, market value of companies or subsidiaries of global companies must be found by different indirect and subjectively biased methods.

The literature on intellectual capital emphasizes the role of knowledge and people in organizational performance. According to the well-known Bontis model of intellectual capital (Bontis 1998) intellectual capital consists of three main components: a) *human capital:* knowledge and skills of employees *;b) structural capital* : all non human stock of codified knowledge in an organization, internal processes, information system, organizational culture, etc. c) *customer/relational capital*: relationships the organization has with its stakeholders, mainly customers. Structural and relational capital are very important for the performance of the company, its competitiveness and problems how to measure the impact of these components on financial indicators of the company are still widely discussed. However, these forms of intangible assets are formed by the general strategy of the company, often not specific to local entities and do not play an important role in the discussion on international taxation issues. It is usually impossible to determine the role of local governments or other host country actors in influencing the costs and value created at the territory of the country..

Human capital is the background of the intellectual capital. Employees´ knowledge, skills, capabilities, attitudes can significantly influence the performance of the company and also the willingness of customers to pay for the products and services of the company. Human capital can convert knowledge into innovation and can contribute to the market value of the company.

Human capital is a complex category including not only knowledge and skills acquired during the formal education, but also personal abilities, health condition, motivation, individual values, etc, that are specific to a particular person. An important component of the human capital is the *tacit knowledge* that could be partially acquired during the formal education, but usually a lot of tacit knowledge is formed by the practical experience in the company (or other companies where the worker had been previously employed), interactions within professional communities and different forms of lifelong learning. Most of these features are difficult to be quantified (Bombiak 2016) and even more difficult is to determine what is the share of local conditions and foreign companies in developing these features. Even if the value created by the local human capital were somehow determined, the "fair" proportion of the employer, employees and local authorities in creating this value is extremely difficult to define. However, host country´s tax revenues depend also – to some degree – on the level of highly skilled workers´ salaries and local authorities are interested in attracting the activities with the higher value added also for taxing reasons (although this is not the most important reason).

Moreover, intellectual capital is extremely mobile and flexible part of assets of a company. It can be quickly moved to other country- according to the global company´s strategy. A proportion of value created in a particular country and under particular jurisdiction often changes. This is significant for tax purposes. Changes in the intellectual capital allocation between countries can be motivated also by other than tax planning reasons. However, tax regimes and labor cost are of crucial importance.

A market-capitalization based approach to valuating intellectual capital does not distinguish between human aspects of this value creating factor (human skills, experience, intrinsic motivation, values, etc.) that is *not owned* by the company, and other conditions (hardware, research equipment, internal and purchased research results, organizational and financial tools to support creativity, etc.) purposefully created and owned by the company.

This distinction is especially important for less developed countries and their argumentation on international taxation issues. They point out that people employed in MNEs subsidiaries are basically of local origin, they were educated in the local educational system and their skills and creativity significantly contribute to improving companies´ profits. A "fair" share of these profits should form a tax basis for local taxation. The tax revenues achieved could be used to finance the investment in the local digital infrastructure which can lower "the digital divide".

On the other hand, companies argue that the increase in the productivity, quality and creativity of the local labor force (which can be of multinational origin and educated in different countries) is due to the general conditions created by the company, material conditions, corporate culture and management system, etc. and proceeds from this activity belong to the investor company. Moreover, a risk assumed by the investor should also be taken into account.

Even if a progress will be achieved in defining and measuring individual components of intellectual capital and their role in the value creating process, uniformity in defining what is "the fair" proportion of individual countries in taxing the increased income, will probably never be achieved. However, it seems that the awareness of the necessity to increase governments´ tax revenues to enable the financing of investment in a digital infrastructure (including also required changes in education system and other problems connected with the digitalization) is strengthening in international community.

Data, user participation and their synergy with intellectual property

This aspect of a new business model is also often discussed in international taxation debates (OECD 2017). However, there is no consensus on whether, and to what extent, this factor should be considered as contributing to a firm's value creation. The degree of user participation may not closely correlate with the degree of a firm's digitalization and other factors are important as well. Moreover, user´s participation can be found, in varying degree, also in more traditional business models and has been supported by the globalization process itself.

Countries and researchers that require additional taxation on using data collected from users, point out that exploiting these data can be unique and very important driver of value creation. Highly digitized companies can monetize this information by different ways (use them in pricing, advertising, etc.) and have a significant comparative advantage in comparison with firms that do not dispone with such data.

On the other hand, opponents of this additional taxation argue that data collected from users are similar to data acquired from any other third party and it is difficult to define an activity to which profit stemming from using these data should be attributed.

The important role that user participation can play is seen in the case of social networks. Without user participation they could not work. However, it is difficult to imagine how taxation principles can be applied to this field.

Fragmentation of the business process and outsourcing some activities.

Steve Sammarting (Sammarting "Great Fragmentation, 2014) proclaims that *The Great Fragmentation is a business survival manifesto for a digital era,* This is the way how businesses try to position themselves and survive under new conditions. On the one hand, this trend can be an opportunity for SMEs and startups to expand and going international. On the other hand, large MNEs implement the fragmentation also internally (within the global concern), radically changing their value chain systems.

International mobility in the mentioned field is very intensive. Activities that were performed in one country at the beginning of the fiscal year can be shifted to other country and jurisdiction to the end of the year. Even if we are able to define the value created (very often it is difficult to measure it) by individual units and flexibly trace the movements in the value chain system, it is very complicated to attribute the "fair share" of taxing rights to individual jurisdictions.

Outsourcing is not a new phenomenon in business strategies. However, the digital era creates new technical tools and massive opportunities for large scale outsourcing activities. If the large company decides to outsource the whole function of the company´s business (e.g.to outsource some "shared activities" in the field of finance and accounting or in the area of HR) and the outsourced company is evaluated just on the basis of its contribution to the consolidated performance of the whole company, no individual financial statements are available. It is very difficult for local jurisdictions to define what vale has been created at their territory. It´s actually impossible to define the tax basis for the locally paid corporate income tax.

However, digitalization can lead also to a tendency of decreasing the fragmentation and reshoring some activities to the resident country. *Reshoring* will lead to shorter, less fragmented value chains and higher geographical concentration of value added. It will primarily affect higher technology industries with intensive global value chain policy. (UNCTAD, WIR 2020).

An important factor that can contribute to changes in the value chains decisions can be the COVID 19 crisis. Pandemic can have severe consequences in impacted areas and geographies, making them inaccessible for an extended period of time. As a component of a business impact analysis, companies identify the chain of activities and functions, along with interdependencies: eg., people, process, technology, data, facilities, third parties. (EY 2020) .These decisions can have significant impact on macroeconomic situation in different countries and the expected tax revenues.

The list of problems (or modifications of old problems) facing the tax policy in the digital era, can be immense. It´s impossible to mention them exhaustively. However, it is important to point out that *digitalization simultaneously provides immense tools and opportunities for government regulation and control* .In principle, governments and international organizations are aware of this fact. However, it seems that the "technological race" between tax regulators and practical tax planning policies of companies will be often connected with some "information asymmetry". Not in the sense that public authorities´ specialists are not able to follow the recent development in IT technology (however, sometimes even this can be a problem), but in the sense of being not able to foresee the companies´ strategy and prepare appropriate tools (e.g. of the big data systems) for effective monitoring and evaluation of particular tax planning measures.

4 Is the discussion on Tax challenges of digitalization relevant also for Slovakia?

Slovak Republic is a small country with an important role of foreign investors and high share of manufacturing in the industrial structure. . Tax revenue from taxing MNEs profits (corporate income taxes), achieved by activities at the Slovak territory, form an important source of funds for Government fiscal policy. Taxation is a topical issue and international competition, that is reflecting also the impact of digitalization, makes the adjustment of the tax system to modern era an imperative also for Slovak authorities.

At the first sight, the impact of digitization is obvious in the field of e-commerce. Although statistically not important (before the COVID-19 pandemy), its share is rapidly growing For the time being, the prices in e-commerce are not officially monitored (Fabo 2018) , but it is assumed that consumers pay in the price also the VAT. As internationally agreed, revenues from this tax are to be attributed to the market country. However, there are several technical problems to be solved in the process of collecting and measuring the VAT in cross-border e-commerce. EU guidelines in this field will be relevant also for Slovakia .Let´s briefly summarize EU approaches in this area.

Since 2015, a simplified system is in place to declare and pay VAT on business-to-consumer (B2C) supplies of telecommunications, broadcasting and electronic (TBE) services in the EU. In December 2017 The European Council adopted the “VAT e-commerce package” (EU 2018) that should facilitate cross-border trade, combat VAT fraud and ensure fair competition for EU businesses. This package includes

- B2C supplies of services other than TBE services
- Intra-EU distance sales of goods
- Distance sales of goods imported from third countries and third territories in consignments of an intrinsic value of maximum EUR 150

The package should be implemented gradually. In 2019 two thresholds should be introduced to simplify VAT obligations for microbusinesses and SMEs. First, an annual turnover threshold of EUR 10 000 for intra-EU cross-border supplies of telecommunications, broadcasting and electronic (TBE). Second, an annual turnover threshold of EUR 100 000 up to which the vendor must only keep one piece of evidence (instead of two) to identify the Member State of the customer.

For the time being, it is difficult to comment, whether thresholds suggested are appropriate for all EU member states and to estimate administrative costs connected with the proposed VAT monitoring system.

In the field of direct taxes, especially the corporate income tax, we can assume that differences in tax rates between countries will exist also in the future and harmonized rules of forming the tax base will be difficult to achieve. What can be the role of the permanent establishment concept and its modifications for Slovakia?

Due to the significant role of manufacturing (and especially automotive industry) in the Slovakia´s industry structure it is logical that foreign investors do have some permanent establishment in the country. Traditional PE concept has been internationally accepted for many years and included into bilateral tax treaties, which is true also for treaties signed by Slovakia. Modifications of the PE concept - as suggested by OECD or EU – are also important for Slovakia Many foreign companies allocated in the country their logistical branches and different auxiliary activities.

Even if foreign investors maintain their physical presence in Slovakia, they are adjusting their business processes in a similar way as anywhere in Europe and increasing digitalization plays a very important role in these adjustments. Many issues mentioned above – e.g. increasing role of auxiliary and supporting processes, multisided business platforms, fragmentation, etc. – are topical also in Slovakia. However, question how to measure them and include into the tax base of corporate income taxes, is not publicly raised in the country.

Let´s mention just one of the aspects of the new business strategy – outsourcing of some “shared services” performed for the whole global company. Recent trends in digitalization make possible such outsourcing of the whole business functions. A growing number of multinational enterprises (MNEs) are reaping advantages from consolidating functions into regional shared service centers (SSCs). Some of the benefits include more efficient transaction processing, better systems integration, more uniform policy administration, improved co-ordination of marketing and better supply chain management. SSCs are helping MNEs to achieve economies of scale and scope across the business units. (Davis, 2005)

Shared services centers are certainly units with high value added activities, employing highly skilled personal and can significantly contribute to the spillover of recent technology within the host country The benefits of this outsourcing to the performance of the global company can be evaluated by the headquarters and top management of the company and reflected in the consolidated financial statements. However, what is the value created by the local personal and potential tax base for the locally paid corporate income tax? This is difficult or even impossible to discover. Hopefully, recent debates about tax challenges of digitalization will contribute also to solving these problems.

A key role of intangible assets in the digitalized business processes is a common tendency for all developed countries and Slovakia is not an exception. However, reporting on the structure of intangibles in accounting documents is very poor [Bernadic 2016] and in Slovakia probably worse than in some other EU countries. In this area there is a lot of problems to be (collectively) solved in connection with the potential CIT tax base harmonization.

Probably, the main reason why a discussion on macroeconomic impact of digitalization (in the field of fiscal policy, possible inflation behavior, etc.) is not developed in Slovakia, is the lack of statistical data on the actual situation. [Fabo 2018]. Sometimes the role of digitalization impact is underestimated, sometimes overestimated by mechanical transfer of tendencies and proportions from large developed countries. A lot can be done in collecting the necessary and reliable statistical data. To acquire these data probably some changes in the accounting legislation will be necessary.

The question of Slovak authorities´ taxing rights regarding the income created at the Slovak territory by foreign investors, is of similar character as in other countries. The role of tangible assets in creating the taxable income is not such complicated and its measurement can be improved by using up-to date IT technology and adjusting some accounting principles.

However, incorporating these new accounting and taxation principles into bilateral tax treaties can be a demanding problem.

On the other hand, measuring intellectual capital, and human capital as its crucial component, as well as measuring the impact of these assets on the taxable income, is a very complex unsolved problem. Even if the impact of the intellectual capital on the company´s performance indicators were somehow determined, this would be true for the multinational company as a whole, not for its individual units. And what is the participation of local environment (including also government policy) and employer (foreign investor) in forming the value of its human capital, is even more difficult to define.

It can be assumed. (although not easily statistically verified) that foreign investors – directly and indirectly – contribute to *increasing the digital competences index of human capital in host countrie*s, This is true also for Slovakia. This impact can be important for the whole digital transformation process in a country. What can be the role of government policy in the mentioned context? Under conditions of digitalization and intensive competition between countries, the crucial role of the government is to create an appropriate digital infrastructure (basically in the area of education and publically financed research) and use also the complex of innovation policy measures to make the country attractive for investment based on a high quality of the human capital, corresponding to the digitalization era. Simultaneously it will be important to adjust and permanently improve tax compliance monitoring (based on up-to-date IT technology) and taxation principles to the new reality, so that tax revenues will keep pace with the actual improvements in business performance and created value. In all the mentioned tasks human capital plays a decisive role.

5 Conclusion

Debates on tax challenges of digitalization are basically going on a conceptual level, attempts to operationalize them are rather scarce. However, lessons that can be learnt from this discussion are rather far reaching. New business models of digital companies that will be further developed and gradually (possibly with some modifications) implemented also by other companies, are based on *knowledge* (explicit and also tacit), knowledge transfer and knowledge sharing in different forms and in global dimensions. On the one hand, new forms of cooperation are invented and implemented. On the other hand, rivalry in different forms is intensified.

Taxation itself is a knowledge based activity - both from the governments taxing authorities´ point of view, and from the point of view of companies that are continuously looking for new ways of tax optimization. The sharing of this knowledge between governments is very important. Moreover, the discussion on the (ir)relevance of the PE concept itself provides a lot of information about the recent approaches of the global business players in the era of digitalization.

Another important theoretical lesson (actually of an interdisciplinary character) is *the role of human capital* - its skills, extreme mobility, new types of motivation, etc. It can be said that a *new type of a knowledge worker* is formed in the digitalized business models (or digitalized organizations in general.). Companies are making use of the new talent grown anywhere in the world and realize that it is important to adjust their HR policies and sometimes even organizational structure and organizational culture to new demands. However, knowledge workers of a new type do not appear automatically. They are to be educated by the formal education system, different forms of training courses in the companies and "learning by doing" and practical experience. The role of governments in supporting the digitalization process within their countries means not only creating the digital infrastructure but also creating such an institutional and economic environment in the country that will motivate new types of knowledge workers to stay in their (home) country or come back after gaining some international experience.

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VI

Thoughts on measuring reverse knowledge transfer contribution

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1 Introduction

The company competences to learn, adapt and respond to the market trends can be the critical success factors of the competitive advantage. From this perspective knowledge deployment is vital not only for the success of entrepreneurship but also for economy and societical future (Hasan/Zhou 2015). Knowledge transfer is used as a means to disseminate knowledge in the corporation which serves delivering multiple benefits and increase company´s performance.

The research used to focus on traditional knowledge transfer from the parent companies to their subsidiaries and from the developed countries to the developing countries. However, internalization and increasing decentralization of research and development supported also reverse knowledge flows which reflected also in the studies uncovering less frequent knowledge transfer from subsidiaries to the parent company and from less developed countries to the developed ones. Reverse „capability transfer is defined as a firm´s replication of internal practices, which are performed in a superior way in some parts of the organization compared to other parts of the organization, and which are superior to internal and external alternative practices" (Schotter/ Bontis 2009:151). „Reverse knowledge transfer from subsidiary to parent [stands for] subsidiary experiences that are transfered to parent companies" (Mudambi et al. 2013:49). An example of reverse knowledge transfer could be product development on the local market that is made available for the other subsidiaries through the parent company.

Some research finds reverse knowledge transfer identical to the traditional one. Other studies suggest that reverse transfer requires richer activities, more frequent personal interactions, parent facilitation and a lot of guidance and effort (Borini et al. 2012). „The political complexities of reverse transfer can be more difficult to overcome than those of forward transfer“ (Chung 2014:229).

This chapter is looking into the effects of knowledge sharing and discussing the respective benefits for the multinational. Frontline located subsidiaries face lucrative business opportunities or individual challenges that drive original developments. Making them available to the corporation can be benefitial for other units in similar situation. Active knowledge exchange is saving resources and helps companies to react faster.

Our illustration of reverse knowledge transfer effect is introduced through two examples from subsidiaries operating in Slovakia. Both come from FMCG environment and stand for commercial entities dedicated to sales and marketing of the products. They were selected for their availability and managers´ willingness to share information about reverse knowledge transfer cases. The information was gathered during semi-structured interview. As the reader will learn later, each subsidiary developed unique managerial solution which was successfully sold to the headquarters and implemented in successing countries.

2 Subsidiaries´ engagement in reverse knowledge transfer

Modern view on the multinationals emphasizes the international network of the organizational units and their opportunities to recieve and contribute to the corporate knowledge base (Coakes 2206, Bezzera et al. 2013, Najafi-Tavani et al. 2014, Andersson et al. 2015). The subsidiaries´ resources form a basis for knowledge exchange and allow each unit to benefit from the heterogenous competences available in the corporation.

According to knowledge generation the subsidiaries can either utilize the existing corporate knowledge, and potentially adapt it if needed, or generate knowledge new to the corporation and create new skills (Mudambi et al. 2013). Development of new competences depends on the mandate that the parent company gives to the subsidiary.

The subsidiaries can have different roles in the corporations: some are supposed to commercialize products and services, the others conduct research and development, some serve to manufacture products etc. „If a subsidiary has only the mandate to distribute the products that have been manufactured elsewhere, it naturally has no mandate to develop new product related capabilities. However, even a sales subsidiary might be able to develop unique marketing capabilities that could be used by the home country operation of the parent company“ (Schotter/Bontis 2009:152).

Based on the contribution of the subsidiaries to corporate knowledge base, resp. use of knowledge from this database following subsdiary types can be defined: (1) units that adapt product to local market needs (implementer subsidiaries), (2) units that exploit MNCs´ technological competencies on a global basis (contributor subsidiaries), and (3) units established to augment or create new technological competencies (innovator subsidiaries, Rabbiosi, 2011:99). Such classification of subsidiaries is introduced also by other authors: Nair et al. (2015:282) build on typology based on geographic scope and autonomy, originaly proposed by Birkinshaw and Morrison (1995): (1) local implementer focusing on local production, hence with lower knowledge contribution for MNC, (2) specialised contributor that usually has expertise in certain specific function and helps in diffusing innovation amongst the parent and other units, and (3) world mandate subsidiaries that operate in strategically important market and have high level of resources and expertise, however do not contribute to MNC´s knowledge base significantly more than local implementers. Ordóñez de Pablos (2006:553) similarly sees four subsidiary roles effecting knowledge transfer process: (1) local innovator takes total responsibility for the creation of a relevant know-how, however with limited competitive use out of the country, (2) integrated player implies a responsibility to create knowledge capable of being used by other subsidiaries, (3) local implementer relies on input flows coming from the head company or the affiliated companies and therefore it creates little knowledge by itself, and (4) global innovator is a source of knowledge for other units.

The subsidiaries can learn from their local context and business relationships (Rabbiosi 2011) and via these interactions co-create new knowledge. The operating environment gives subsidiary an access to large stock of ideas. The relationships of subsidiary with suppliers, customers, lead users, outside expert or consultants, universities enhance novelty and stimulate subsidiary innovations so that subsidiaries are able to draw upon the existing knowledge pool in the local environment (Piscitello/Rabiosi 2006:1). Having generated new knowledge subsidiaries get ready to contribute to evolve MNC´s existing products/services, or even develop new services, technologies, or products (Najafi-Tavini et al. 2011:466).

The subsidiaries that are more active in innovations have more knowledge that could be transfered to the parent company. The amount of knowledge which is owned by the subsidiary is effecting reverse knowledge transfer. Should the subsidiary own only little knowledge, it can benefit very little from not sharing this knowledge. Such subsidiaries are usually engaging in reverse knowledge transfer although their value for the corporation is low. Following the increase of knowledge amount the subsidiary gains more attention from the parent company and the intensity of reverse knowledge transfer will go up. When the subsidiary manages to further grow its knowledge base, the interest from the parent company will again strengthen but it could happen that the subsidiary will realize its power in the corporation. Knowledge transfer will be equivalent to decreasing subsidiary power and control over its knowledge. However, in the context of long-term relations we can also expect a cooperative approach – the subsidiary which passes on its own knowledge in the longer run is strengthening its role in the corporation (Mudambi et al. 2013).

The transfer of knowledge that is embeded in employee experience and skills and highly tacit knowledge is difficult to transfer and often requires physical presence of the parties, with the cost impact. The cost related to knowledge exchange is relevant to both parties and partners expect to share it fairly. The subsidiaries of the multinational are willing to invest into corporate membership and usually allocate resources into knowledge flows (Chen et al. 2011). Further, it has been proved that more complex knowledge, that is assumed to be more costly to transfer, is exchanged more often (Nair et al., 2015).

3 Knowledge transfer effects

Integration of knowledge into the products and technologies helps corporations sustain their competitive advantage, a skill especially needed on highly competitive markets (Ling et al. 2009). Managing knowledge circulation process has a positive impact on sustainable performance and competitivness of the organizations (McKeen et al. 2006).

Knowledge transfer is often thought to improve organizational performance since the units get new information about their customers, trade partners, legal environment, elements of external and internal environment and can respond to the changes faster. Knowledge transfer accelarates the innovation rate thanks to ideas sharing, stimulation of new practices and helps to assess the commercial potential of the innovations (Van Wijk et al. 2008). „Non-financial [] factors such as acquistion of patents, increasing competitive power, and employees´ productivity [stronger innovative capacity] are the key determinants effecting overall organizational performance," (Rhodes et al. 2008:96) and can be acquired and/or enhanced through knowledge transfer.

The positive impact of knowledge management on company performance, on various non-financial indicators e.g. competitivness, customer relationship, productivity, share of market, higher rate of innovation, strengthening consumers´ satisfaction, is declared by many academic researches. According to some authors knowledge management practices reflect also in profitability increase, see figure 1.

Figure 1 *Performance indicators that can be improved by knowledge management practice*

Indicators of Knowledge management performance	Performance indicators	Source
Non-financial indicators	• Innovations, efficient use of resources, productivity, learning ability, knowledge re-allocation ability, product quality, faster market responsivness, anticipation of problems, competitive advantage	McKeen et al. (2006) Darroch (2005)
Financial indicators	• Profit	Darroch (2005)
Models	Framework combining three group factors • Financial performance – ROA, ROE, ROI • Customer satisfaction – product leadership, firm image perception • Operational effectivness – operational excellence, i.e. responsivness to customers and improvements in productivity	Ping-Ju Wu et al. (2015)
	Knowledge management effectivness measured through input-output ratio • Inputs – number of employees, research and development expenses, administrative expenses, advertising expenses • Outputs – net revenues, value of intelectual property	Chang et al. (2011)
	• Organizational competitivness – product leadership, customer intimacy (understanding and retaining customers), operational execellence – consequently partial improvement of financial performance	McKeen et al. (2006)
	Knowledge management performance index • Knowledge circulation process (creation, accumulation, sharing, utilization, internalization) in relation to stock price, price earning ratio, research and development expenditure	Lee et al. (2005)

Successful knowledge transfer increases the level of knowledge of the recipient and translates into implementation of new techniques and processes. The acquired competences can materialize on the market. New knowledge can be intengrated into design, manufacturing process, system of operations, sales and distribution systems, economy and management (Chen et al. 2011).

Succesful knowledge sharing can be measured through the level of assimilation of new knowledge; however the challenge is the isolation of the other factors that could have impacted the change (Easterby-Smith et al. 2008). Measuring tacit knowledge effect is more difficult than the effects of explicit knowledge. Another issue of measuring the outcomes of knowledge transfer is the fact that knowledge is often disseminated throughout the organization (Argote/Ingram 2000).

Andersson et al. (2015) measured the effectivness of knowledge transfer via quantity of shared knowledge and the cost of sharing. It proved that companies put focus on either dimension, i.e. either they try to minize the cost for knowledge sharing or they attempt to maximize the amount of exchanged knowledge; the preference depends on management decision or organizational objectives. Tseng (2015:121) similarly judges transfer cost and compares it to the cost of gaining knowledge from the external environment: „If the cost of intra-firm knowledge flows is lower than that of inter-firms knowledge flows, then the MNC will directly infuse knowledge into its subsidiaries.“

In case of reverse knowledge transfer we believe to see a multiplicative benefit even accelarated by the speed of initiative introduction in the other unit. Notwithstanding the local context it could be assumed that knowledge that is delivering positive effect on the performance of the original subsidiary will result in similiar positives in the unit, that it is transferred into. This assumption we will discuss in the next part of the chapter.

4 Reverse knowledge transfer cases

The effect of knowledge transfer is going to be illustraded on the real cases from two FMCG companies. Their managers outlined the impact of the initiatives developed in Slovakia and confirmed that they were deployed by further markets. In both MNCs knowledge sharing is enforced by organization and formal control mechanisms that ensure the transfer of best practices to the other subsidiaries with potential room for implementation. The transfer is always facilitated by headquarters which have an overview of situation within individual markets.

Integrated information system in JTI Slovak Republic

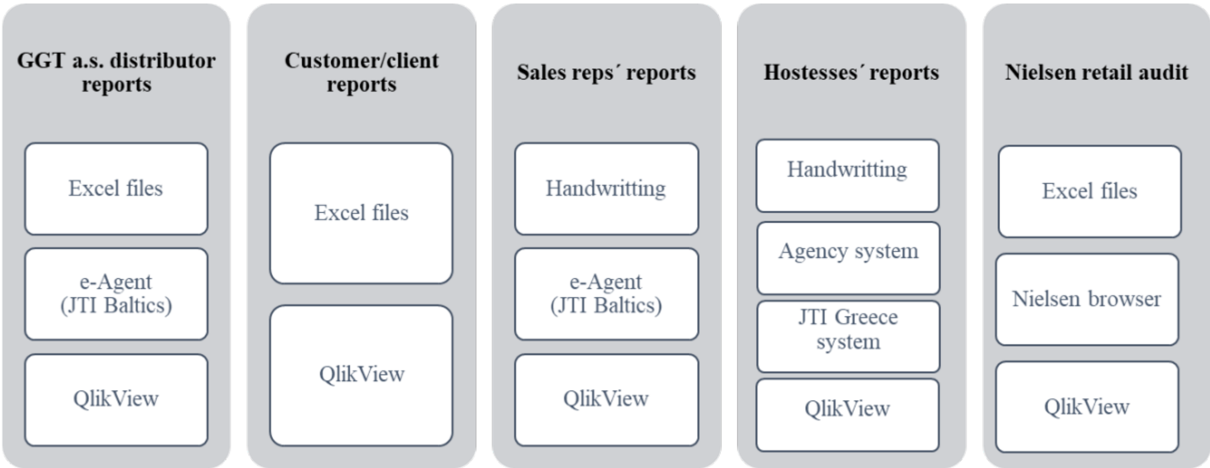
JTI Slovak Republic, s.r.o. (JTI SR) was established in 2004 as a subsidiary of the global tobacco MNC and has a role of marketing entity: neither it produces nor it sells. The import of JTI products to Slovak market is executed by the third party Tabakoland, a.s. and distribution of the products is done through GGT a.s. while JTI SR is responsible for sales and execution at the point of sales, i.e. the sales and marketing programms towards wholesalers and retailers. JTI SR enjoys a growing trend and its objective is to sustain the momentum to become more important contributor to global JTI sales.

In the past JTI SR had limited resources and many of the statistics and reports were prepared in the excel tables or even pen & paper. In the course of time the subsidiary moved to more advanced information system; the systems developed in the other subsidiaries or by trade partners were introduced. Change to automated reporting allowed JTI SR employees to spend less time on data processing and more time on data reading and increase the quality of planning. Having multiple data systems and sources, JTI SR got an idea to integrate all of the information to one data warehouse and introduced QlikView, a business solution developed in cooperation with local provider EMARK.

QlikView is integrating very complex commercial information pieces about sales from and GGT who sells primarily to wholesale level and own chain of tobacconist, further data from various customers who provide with the sales details from the specific retailer chains or even outlets. Additionally, data from sales reps´ team about performance and competitive situation in the specific outlets as well as data gathered by hostesses running promotions is included in

QlikView. Finally, QlikView adds retail audit data from renowed research agency Nielsen that provide with information about market trends, market shares, pricing etc. The overview of data, that is used in JTI SR and the development of the respective ways of its processing towards QlikView system is captured in figure 2.

Figure 2 *Overview of data used in JTI SR and the development of the respective systems*



QlikView is used as the key knowledge base for strategic commercial decisions as well as for monitoring and planning of sales and marketing initiatives. It allows users to get the information fast and easily, conduct analysis and connect the commercial end points. Combining data from different sources is the key advantage of the system and stands for the system´s biggest advantage; the user can quickly analyze performance of the specific outlets or key accounts against general trends, the effect of sales and marketing initiatives is easy to define by simple comparing performance of the key accounts with and without the respective initiative.
Using QlikView validated some effects of commercial initiatives and made JTI SR confident rolling them to the market. Launch of limited edition packs delivers upto 30% volume increase that is sufficient to offset the cost for the development and marketing support. Once limited edition is introduced on key products with higher rotation and volume and is sold on the market for four months the project result is delivering profit of 50 tsd € and more. QlikView suggests that smokers of premium brands are loyal thus sensitive price increase of 0,10€ esp. when combined with the tax change translates to minor volume decline that is immediately offset by pricing effect; final impact on profitability is strongly positive. On a small brand, typical for premium segment, the effect exceeds 150 tsd € in a year. Another case analyzed by QlickView is packsize decrease from the past, from 20 sticks to 19 sticks. Negative impact on volume of

10% was exceeded by rise of profit as the company kept end user price while decreasing the production cost due to material saving (Saloňová 2017).

The mentioned examples of commercial initiatives supported by QlikView reflect in the profit of thousands of euro which suggests that the cost of launching QlikView returned within one year. QlikView makes the life of JTI managers easier, the analysis are done faster which assumes that company could have had skipped some of the projects, resp. would have had introduced some at later stage. This indicates clear QlikView contribution even though it can´t be exactly defined as we don´t know which projects would only run thanks to QlikView and further we also miss data to calculate the effects in long-run. Nevertheless, the indicated value of QlikView is described; QlikView is supporting commercial decisions and planning which leads often to increase of profitability.

Following the effects in Slovakia, JTI headquarters recognized QlikView positives and recommeded system to several other entities for deployment. QlikView was presented to few other markets and Georgia and Denmark finally introduced it while localizing to their context. Basically, they foresaw the benefits of QlikView functionalities that allow data processing on more advanced level which is needed for successful management of commercial function (Saloňová 2017).

Logistics outsourcing in Maresi

Maresi is a member of Vivatis holding which shares are owned by Raiffeisen group. Maresi is operating in five countries including Slovakia. Maresi is a foodbroker company, that provides services to the owners of FMCG brands; their commercialization on local markets is executed by Maresi. The brand owners don´t need to operate an entity in the local market, since Maresi foodbroker is able to manage all sales and marketing activities: ensure distribution to key wholesalers and retailers, close agreements with key accounts to conduct sales, support products´ sell out by appropriated marketing, do instore merchandising and mitigate risk of foreign.

Maresi Slovakia (Maresi SK) used to distribute and market on local market also Ferrero, Storck, Chipita and Haribo, substantial turnover contributors, and very much suffered after these brands terminated the cooperation with Maresi SK. Maresi SK lost significant portion of revenues and recorded a huge cost burden; the share of logistic cost on nett nett income (sales deducted by bonuses) rose from 3,4% to 7,8% and share of HR cost on nett nett income rose from 7% to 13%.

These records triggered the countermeasurement within logistic function – potential of logistics was not utilized and the function presented significant fixed cost. The cost height was determined by the fact that company owned a warehouse in capital Bratislava, 4 distribution vehicles and employed 4 drivers and 4 stock keepers in warehouse. For these reasons Maresi SK focused on logistic cost reduction.

Initially, multiple optimization measurements were introduced: stop of the unit sale, set up of minimal order value of 200€, split of Slovak region to 6 areas to optimize the logistic routes, set up of logistic cycles to achieve better efficiency, launch of EDI that allowed to issue invoices faster and translated to improved cash flow. The mentioned initiatives decreased the share of logistic cost down to 6,9%, less by 1 ppt., however it materialized only after one year. In the first year company terminated the contracts with logistic employees and sold out the warehouse and vans, but at some reimbursement and compensations.

In parallel logistic audit was prepared and revealed the most efficient alternative: complete function outsourcing. Maresi SK selected local logistic partner with strategic position close to key customers, with advanced logistic systems which are able to generate additional savings and with much bigger capacity that is also used by other vendors and reflect into economies of scale. After the full logistics function got outsourced the share of cost went down to 5% (Slobodová 2017).

Figuer 3 shows price list provided by selected logistics supplier.

Figure 3 *Prices of logistic duties from external provider to Maresi SK*

Area	Price for delivery upto 100 kg (€)	Tariffs for delivery of extra weight (€)						
		100-500 kg	501-1000 kg	1001-2000 kg	2001-3000 kg	3001-5000 kg	5001-8000 kg	above 8000 kg
Bratislava	6,61	0,055	0,048	0,038	0,033	0,029	0,025	0,022
Bratislava district	6,97	0,058	0,051	0,041	0,037	0,032	0,028	0,024
Trenčín / Nitra district	8,23	0,078	0,076	0,058	0,041	0,036	0,032	0,027
Žilina / Banská Bystrica district	11,39	0,096	0,094	0,073	0,056	0,049	0,043	0,037
Prešov / Košice district	14,44	0,133	0,0115	0,1	0,073	0,063	0,055	0,048
Notes:								
* prices are fixed, subject to change only in case diesel price per liter exceeds 1,34 €								
* road tax of 9% will be counted on top of above mentioned prices								

Logistic service	Unit	Unit price (€)
Administrative work	Monthly fee	663,878
Packaging	Item	0
Commisioning	Carton	0,083
Pallet In	Pallet	0,996
Pallet Out	Pallet	1,461
Warehousing	Pallet/day	0,194

As the prices show part of the logistic cost is fixed, part of the cost is variable and is determined by the quantity and place of delivery and therefore the saving on logistic cost should not be determined in absolute value. Nevertheless, when calculating cost per ton of goods, the saving is approx. by 30% compared to when logistic function was run Maresi SK. Additional saving comes from no requirements on logistic headcount and tangible assets (warehouse, vehicles). All in all, moving to external logistic provider was a stepchange for Maresi SK with substantial impact on operational efficiency, clearly validated by cost reduction.

The logistic savings achieved in Slovakia primarily thanks to function outsourcing tempted the headquarters who kicked-off the audit of logistics in all other countries. The project was lead by Slovak manager, who had already experience and knowledge about subject. The analysis outcome was in favour of logistic outplacement in all countries, including Austria where production site for Maresi brands is located and two warehouses were operating. Step by step all Maresi subsidiaries gradually outsourced the logistic and sometimes introduced also some further optimization measurement that were originaly developed in Slovakia in order to optimize the functional cost (Slobodová 2017).

5 Discussion about measuring effects of knowledge transfer

Our two case studies demonstrate that subsidiaries operating in Slovakia generate new knowledge, even the smaller subsidiaries dedicated to product commercialization search for solutions which enhance their operational performance. New knowledge is generated when the organization lacks adequate practices, a process is driven either by subsidiary ambition to grow or by its need to solve a specific problem. Although the introduced examples of original initiatives are rooted in different business function, they both aimed at elements critical to the whole operation: logistic performance with the potential to stabilize overall results and enhanced IT system supporting sales and marketing management to meet growing ambition.

The value of QlikView developed in JTI SR refers to consolidation and combination of all sales and marketing data vital for running commercial entity. This data is used daily to track sales performance, impact of marketing program, threat of competitive activities. More profound analysis of sales development, market trends and marketing performance are prepared during planning sessions when strategic directions are comfirmed. It is highly appreciated if input information is available and easy to process, an advantage delivered by QlikView. Thanks to its benefits QlikView is seen as optimal system to support development of tactical activities as well as strategic planning which can translate to increased competitivness on the market.

For quantifying QlikView effects we indicated few projects that were introduced to Slovak market. Launch of limited edition pack generated incremental EBITA of 50 tsd €, controlled price increase of premium brand 150 tsd € in one year, and packsize decrease translated to incremental profit too. These initiatives are illustrative examples of projects supported by QlikView, although in general it is challenge to distinguish between projects developed solely thanks to QlikView. One should understand QlikView as a system supporting business decisions which helps to assess the alternatives, model scenarios, analyse the potential risk and mentioned financial impact of illustrative examples need to be considered as indicative effects of commercial projects supported by the system.

The other units that deployed QlikView after Slovakia are assumed to consume similiar benefits; the system is supposed to support their decision making process in sales and marketing function. It can be expected that various commercial initiatives of the subsequent subsidiaries were backed by QlikView data. Compared to originating entity the successors enjoy the benefit of immediate availability and very limited need to invest into system development. Thus, for the other organization units QlikView does not provide pure organic system benefits which translate to improved commercial competitivness but accepting QlikView from Slovakia was also very efficient solution without designing requirements.

The challenging situation Maresi SK was in triggered multiple optimization initiatives with the biggest impact comming from logistic outsourcing. Hiring third party for providing logistic services decreased direct logistic cost, recalculated per tone of goods saving of 30%. Further saving was delivered by eliminating logistic staff and assets previosly kept in the company. Using specialist instead of keeping the function in-house allowed to run logistic duties more effectively which strengthened Maresi SK operational performance.

The subsequent subsidiaries could not take ready system like in JTI, the object of knowledge transfer was the idea of transfering logistic to the third party, which still needed to be assessed locally. The outcome of analysis was in favour logistic outsourcing in each Maresi market which indicates substantial cost reduction opportunities for the whole MNC. Although other subsidiaries did not lose the revenues, a starting point for cost optimization exercise in Maresi SK, the decrease of logistic cost that could be achieved following the function outsourcing was valuable contribution to financial performance.

Defining the effect of new knowledge implementation can be quite challenging task. Some initiatives translate to quantifieable results, some results are more appropriate to be described by qualitative statements. As we saw, even a saving on logistic cost should not be counted in absolute value, as overall cost very much depends on distributed quantity of goods and respective distances. More adequate indication logistic cost is comparison of cost per tone of goods or ratio agains revenues. In case of new commercial IT the direct benefit is delivery of more confident and quicker decisions in sales and marketing which strengthen company competitivness and translate to financial KPIs. However, in case of QlikView it is hard to select all relevant measurements which introduction to the market was driven solely by the system.

Once we agree on impact of knowledge driven initiatives in the home market operation, we may start our thinking about the effect on transfering subsidiaries. Some initiatives are at the readiness stage, which means they can be immediately taken by next units, the other initiatives require further localization. In case of ready made initiative the benefit for successor is substantial as the organization can materialize the positives of solution immediately and can enjoy them without development cost. However, when local development is necessary, the unit needs to spend time and resources prior to knowledge utilization, a process that can be quite smooth considering the learnings from originating market were shared. Best practice and knowledge sharing is the approach that is welcome in MNCs as it supports improvements accros the organization. To know exactly the total contributions, we would need information inputs on them from all implementing entities though.

In conclusion, effective managment of knowledge transfer helps MNCs to increase their competitive advantage through combining local knowledge, and technological and managerial know-how. Subsidiaries have huge room to generate new practices, innovate products, improve current way of working and pass it to the parent companies. Knowledge sharing becomes then very effective tool to gain new solutions – utilization of knowledge from the other subsidiaries is nowadays common practice that further supports efficiency and synergies. Inflow of knowledge avoids duplicate development cost. Reverse transfer further helps coordinate global strategy, align the product offer, drive technological development, monitor the development of the subsidiaries, align knowledge processes and mitigate the risk thanks to earlier implementation elsewhere. Finally, subsidiary that is sharing its know-how increases its relevance in the corporation and can gain headquarters´ acknowledgement.

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VII

Methods of measuring the knowledge impact on organizational performance

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1 Individual knowledge and performance

"It is not what a single bee knows that matters. It is the collective wisdom of the bee colony that brings the results." Jozef Šimúth Sr.

As the business is becoming more dependent on knowledge and information exchange, organizations need to focus on the possible ways of measuring knowledge to be able to evaluate human capital value. From a business point of view, knowledge has become one of the most important factors of production being added to labor, land, and capital. According to McIver and Wanng (2016) managing knowledge is a key factor in facilitating innovation and product development (Pitt and MacVaugh, 2008), increasing productivity (Wiig and Jooste, 2003), and enhancing operational efficacy (Hult *et al.*, 2004). Understanding knowledge involved in organizational work has become critical to organizations' existence and success. Many knowledge management researchers and experts argue that the key and most complicated element is actually the ability to measure knowledge in organizations. With a minor exaggeration some of them claim that what one cannot measure does not exist. In this chapter, we attempt to describe the key psychological variables related to knowledge acquisition, retention and retrieval. Then we indicate possible ways to measure knowledge and issues related to measuring individual knowledge. Then we introduce a so-called beehive approach to organization knowledge measurement. This approach focuses on measuring collective knowledge that gets visible on organizational performance instead of measuring individual knowledge of each organization's member.

There is a belief that it is natural for businessmen to rely on data that come from reliable measurements in order to be able to plan and decide appropriately. As we mentioned knowledge becomes a vital factor for business success. Therefore, managers are asking the knowledge management experts to provide them with reliable ways to measure the knowledge in their organization. However, this is not as simple as they may wish. The first obstacle comes when we look at the definition of knowledge. A dictionary entry defines knowledge in several ways:

a) the fact or condition of knowing something with familiarity gained through experience or association,
b) acquaintance with or understanding of a science, art, or technique,
c) the fact or condition of being aware of something,
d) the range of one's information or understanding, answered to the best of one's *knowledge* (online Merriam-webster dictionary, 2020)

Henriques (2013) defines knowledge as an awareness of or familiarity with various objects, events, ideas, or ways of doing things. Moreover, Ipe (2003) incorporates other variables stating that knowledge is context specific and relational because it is based on experiences and its formation and shaping.

At least to some extent, the knowledge depends in part on how a person comes to understand things. Hvorecky et all (2013) explains this on following example, the perceptual, cognitive contextual organizations allow a person to experience and understand *a vase* on the desk in a specific way. Different perceptual or cognitive contextual organizations would result in a different reality. This context depends on various factors such as the cultural background, empirical experience, perceptual sets, stereotypes etc. There are many factors that influence the way we experience and learn about the world a thus form knowledge.

Svejby (1997) focuses in the definition on a behavioral variable of representing knowledge. He defines knowledge as "a capacity to act". According to him, this makes the important distinction between the behavioral potential, which cannot be directly observed, and the observable performance or behavior. The failure to distinguish between knowledge and behavior prevents the formulation of precise questions about the full process by which individuals acquire, retain and manage knowledge to perform tasks safely, effectively and at a high-quality level. Quine (1987) includes a motivational variable by raising a question - where is the boundary between being certain enough and not being certain enough? From a purely practical perspective, a person must be confident enough so that he/she will use the knowledge to make decisions, solve problems and act. The level of confidence required to be qualified as knowledge may be different depending upon the utility or importance of the consequences. In other words, it means evaluation of the possible gains if the action is correct or the possible costs if the action is incorrect.

Another variable when discussing the knowledge is the variety of knowledge structures. To fully grasp the concept of knowledge, we need to consider the various types of knowledge such as declarative and procedural. The distinction between declarative and procedural knowledge is basically the difference between knowing that and knowing how. Another important distinction similar to the previous one though is between explicit and tacit knowledge. Explicit knowledge consists of well-structured facts, rules, relationships and policies. Explicit knowledge can be stored on paper, media or by other means. Hvorecky et al. (2013) states that computer programs represent the most advanced forms of explicit knowledge today. Knowledge is transformed into the abstract machines capable of getting data from their environment, remembering and processing them and producing results. Mathematical and chemical formulas, recipes and operational instructions – all are examples of explicit knowledge. This part of knowledge is quite easy to measure and manage.

By contrast, tacit knowledge is stored in human memory only. A person can register its existence only when it is manifested in behavior. Hvorecky et al (2013) example of tacit knowledge on interpretation of statistical data. People are likely to read the same data in different ways depending on their experience, familiarity with the environment, emotions, political views, etc. Even if some guidelines on interpreting statistical data can be proposed, there is no universal method, and the result depends significantly on its interpreter. Sporadically, a person might not be aware of processing a piece of tacit knowledge. Then the activity may seem to be random and the outcome as good luck. One important aspect of a tacit knowledge is that it is large proportion of our knowledge. Ozhorn et al estimate ratio 90:10 in favor of tacit knowledge. This is important to bear in mind as we assume that there are limited possibilities for measuring tacit knowledge.

Similarly, Kahneman (2011) describes two systems in using and representing knowledge. He refers to System 1 as to a system that operates automatically and quickly with little or no effort and no sense of voluntary control. He describes the system 2 as a system that allocates attention to the effortful mental activities that demand it, including complex computations. This is similar to the explicit knowledge usage. He claims that the operations of System 2 are often associated with the subjective experience of agency, choice and concentration. The system 2 is the conscious reasoning and we believe we use system 2 to makes the choices and decide what to think about and what to do. However, we agree with Kahneman, that although the system 2 believes itself to be where the action is, the system 1 is the one we use the most frequently.

Another important factor is the process of remembering and recalling information from memory. For knowledge measurement, it is crucial to understand the process of storing and recalling knowledge. We can say that memory is any sign that learning has remained over time. It is individual's ability to store and retrieve knowledge. Atkinson and Schiffrin (1968) described three-stage model of memory which includes

a) sensory memory,
b) short-term memory,
c) long-term memory.

There are two processes involved in remembering in all three stages: the conscious effortful processing and the automatic unconscious processing. For knowledge measurement, knowing how information is stored seems to be the most crucial aspect. There are actually two types of long-term memory explicit or declarative memory and implicit or procedural memory.

Explicit memory refers to facts and experiences that one can consciously know and declare. When we remember our vacation in France or recognize that some words occurred in a recent list, these are instances of explicit memory. In cases of explicit retention, people respond to a direct request for information about their past, and such tests are called explicit memory tests. Implicit memory encompasses learning an action while the individual does not know or declare what s/he knows. Implicit memory involves learning an action while the individual does not know or declare what s/he knows. The explicit memory is similar in description to the concept of explicit knowledge and the implicit memory to the tacit knowledge. We assume that the System 1 uses mainly the tacit knowledge stored in the implicit memory. While the slow system 2 uses the explicit knowledge stored in explicit memory (Henry et al 2017).

Another issue related to memory is the retrieval process. If we are to measure the knowledge of an individual, we are basically attempting to measure the information stored mainly in the explicit memory. However, the process of information retrieval is not straightforward. It is affected by retrieval cues, context effect, current emotional state, physical decay on the neural level, retroactive and proactive interference between newly learnt information and already stored information. An important factor is also that our memory actually functions as an active agent in constructing memories either by changing parts of information or even creating new

memories without any learning process. This was shown in extensive research by E. Loftus. She found that human memory does not only record information, but it also creates new memories thus it is a dynamic process that changes continually. These studies show that when people who witness an event are later exposed to new and misleading information about it, their recollections often become distorted (Loftus, 1997).
The mood of a person is another factor that can play an important role in the information recall of explicit and implicit memories. Bower (1981) based on his experiments states, that people attend to and learn more about events that match their emotional state at the moment of learning. Moreover, he discovered mood-state-dependent retention. This means that people recall an information better if they are experiencing during the recall the original emotion they experienced during learning.

These were only brief explanations of some psychological variables related to knowledge processing, storage and retrieval. These examples clearly show that not only is part of individual's knowledge hard to access, it can vary depending on external situation and context. Therefore, measuring knowledge is a very challenging task. We basically attempt to measure a very dynamic phenomenon. Even if we agree that we can measure the level of knowledge at a given moment, we are still not getting the full picture as we are not able to access the vast amount knowledge that the person is possessing.

Knowledge management experts tend to focus on the knowledge or on the processes that are involved in knowledge sharing. It means, the knowledge measurement focuses on the results of knowledge use instead of measuring knowledge as such. Measuring explicit knowledge is obviously not as challenging as measuring tacit knowledge. Knowledge tests as used at schools can be quite accurate if we manage to eliminate factors such as context effect and test anxiety. For measuring tacit knowledge, classic knowledge test would be inefficient for the reasons we described earlier.

According to Grant (1996), identifying and forming the characteristics of knowledge that have critical implications for management will provide more value than resolving the long-lasting debate on what knowledge is. There have been attempts to develop a reliable and valid scale for measuring the knowledge. One such method is the Knowledge-in-practice (KIP). KIP framework was developed by McIver et al. (2013) and identifies knowledge characteristics which have critical implications for management. McIver et al. (2013) define KIP as the information and know-how involved in the sequences, routines, capabilities or activity systems in an organization. KIP is conceptualized on a continuum of two dimensions: tacitness and learnability. Tacitness is defined as the degree to which the know-how involved in an practice is unobservable, difficult to teach, unspecifiable, and/or highly embedded in the setting (McIver et al., 2013). Learnability is defined as the type and amount of effort, study, accumulated comprehension and expertise that is involved in understanding the information and know-how for accomplishing work practices.

The learnability and tacitness of KIP provide a way to differentiate the essential knowledge characteristics. When these dimensions are combined, they make according to McIver and Wang (2016) four practice types:

1. accepted information (high learnability, low tacitness);
2. apprenticed know-how (high learnability, high tacitness);
3. accumulated information (low learnability, low tacitness); and
4. talent and intuitive know-how (low learnability, high tacitness).

These dimensions are measured by a survey about work activities and characteristics of the department in which a worker works. These surveys are designed specifically for each work context separately. An example of the knowledge measurement at a hospital unit can be found in McIver and Wang (2016).

There are human resources management practices that try to measure the tacit part of the knowledge. Matoskova (2016) lists three approaches to measuring tacit knowledge which are including human resources management activities:

1) monitoring the performance of individuals during real or simulated work situations. The primary focus is on manifestation of knowledge in working behavior,
2) situational judgement test,
3) questionnaire evaluating behavior.

There are basically 6 approaches to measuring individual work performance. The first one is called Management by objectives (MBO). This method is based on setting clear goals using the SMART method for the person for a given period (quarterly, semi-annually, annually). At the end of the period the results in meeting the goals are measured.

The second measurement is 360-degree feedback. It is a multidimensional method measuring given criteria on a Likert scale related to persons work performance based on the feedback from managers, colleagues, subordinates, customers, self-evaluation and others if relevant for a given job.

The third method is Assessment Center Method. A person is observed during in-basket exercises, informal discussions, fact-finding exercises, decision-making problems, role-play, and other exercises that ensure success in a job. The observers are usually job experts, psychologists and supervisors. The positive aspect of this method is that it not only measures knowledge manifested in behavior but it can also predict future performance based on demonstrated knowledge.

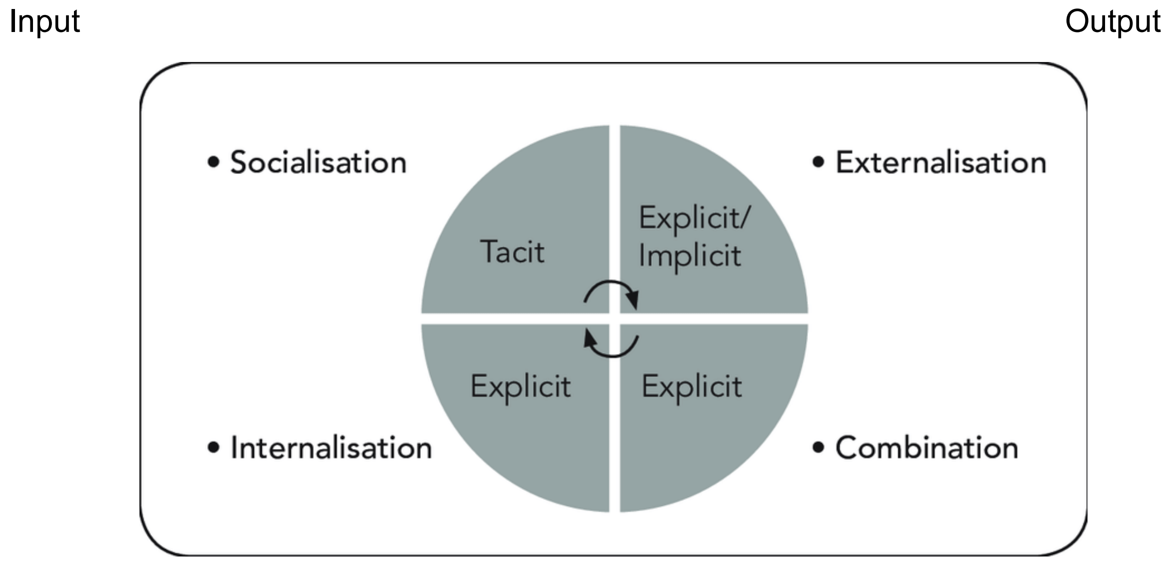
The fourth method is Behaviorally Anchored Rating scale (BARS). The measurement is based on comparison of persons job behavior with specific behavioral examples that are anchored to numerical ratings. Each behavior level on the scale is anchored by multiple statements describing common behaviors that the person usually performs. The first step in the BARS is creating critical incidents and transforming them into a common format. The critical incidents are then randomized and measured for efficacy.

The fifth method includes the psychological appraisals mainly focused person's psychological inner context of using the knowledge such as interpersonal skills, cognitive abilities, leaderships skills, personality, emotional intelligence, cognitive and decision-making styles, etc. This knowledge which is prevailingly tacit is measured by psychological tests such as MBTI, EQ tests, Cognitive Style Indicator, Situational Judgement Test, and many other tests. Other psychological measuring methods are in-depth interviews, discussions, focus groups etc. These measurements are assessed by trained psychologists. Sometimes this method is used as a part of complex Assessment Center Method.

The sixth method is the Human Resource Accounting method which assesses the employee's manifestation of knowledge in job performance reflected in monetary benefits to the company. This method simply compares the costs of employee to company and monetary benefits (contribution) the company gains from activities of that particular employee. This method is also used as calculation of Return on Investment (ROI). This method does not take into account any potential levels of tacit knowledge which have not been manifested in the given period of measurement. Thus, it does not actually measure the real knowledge possessed by the person.

Actually, all these methods have several limitations. They are either depending only on self-reporting or subjective observation or are too complex requiring professional training and expertise. Each of the method is only partially applicable to all possible organizational contexts. One way of mapping the knowledge within the organization is the SECI model. The model does not directly serve as a knowledge measurement method. The model explains the dynamic relationship between tacit and explicit knowledge. According to the authors Nonaka and

Takeuchi (1996) it is a model of knowledge creation that explains how tacit and explicit knowledge are transformed into knowledge that becomes part of an organization.



Picture 1. SECI model. (Source: Nonaka & Takeuchi, 1996)

According to Hvorecky (2013), conveyors of tacit knowledge interact with conveyors of (often different) tacit knowledge during Socialization. It is achieved by interpersonal communication. The authors consider this as the most traditional knowledge transfer between humans. To gain a knowledge that is independent from a person, people try to express their understanding of the world e.g. from interpersonal communication in a commonly accepted system using diverse forms of Externalization. These externalizations or presentations of knowledge are in form of text, formulas, graphs, numbers etc. This creates a basis for knowledge distribution. The receiver of the formally noted knowledge can process this knowledge and this can lead to creation of new piece of knowledge in combining with existing knowledge of the receiver. This is called Combination stage. Then through internalization the new piece of knowledge becomes an integral part of individual knowledge (Hvorecky et al, 2013).

From this explanation, of the SECI model, it becomes clear that the knowledge can become measurable when the organization manages the processes of Socialization, Externalization, Combination and Internalization properly. Only knowledge that is communicated and then coded can really be measured. Hvorecky et al (2013) describes few practices for every stage. The activities in the socialization stage are usually storytelling, discussion, listening to other opinions, teaching and training, brain storming, brainwriting, mind mapping etc. The Externalization stage includes capturing one's ideas core, formalization of ideas, notations, demonstrating skills, posing "right" questions. The authors include activities such as practicing a new activity, learning a new formal notation, understanding potential usefulness of information. In the Combination stage, they mention lateral thinking, creating analogies, identification of a new piece of knowledge.1Through Internalization, the new piece of knowledge becomes an integral part of our individual knowledge ready for its future application.

One of the possibilities to overcome the issues with measuring the psychological variables is to look at the overall organization performance. We call this a beehive approach to knowledge management. In other words, in an organization individual knowledge is hard to measure like it is hard to measure individual bee knowledge and performance. It is the wisdom of the bee colony and not of the individual bee that matters in the end.

Therefore, in the next part we will focus on a description of the some of the best valued and most comprehensive methods that can be used to evaluate the impact of knowledge on the organizational performance.

2 Organizational knowledge and Performance

The key component of our beehive approach is to enable the transformation of individual knowledge into organizational knowledge. Thanks to various stages of knowledge processing - creating, accumulating, organizing, and utilizing – organizations can improve their organizational performance (Rasula, J and Co., 2012).

Organizational performance could be evaluated with various methods, including the financial or non-financial indicators. Further, we describe several methods that in various extents include financial and non-financial indicators. The financial measures such as of budgets, assets, operations, products, services, markets and human resources are crucial in influencing the organizational success (Riley, S. M, 2017). However, the organizational performance should include more than just the return on investment calculation and to be considered more dimensionally. Organization performance has been the most important matter for every company. It has been crucial for managers to find out which factors affect the performance of their organization in order for them to apply necessary measures to initiate them. If an organization wants to survive and make profits in information age, it should use performance measurement systems that are derived from their strategies and capabilities (Ekström, M., & Hammarlund, L., 2020).

There are three sides of organizational performance:

(a) a goal approach where the performance is evaluated by reaching a goal;
(b) a systems resource approach where the ability to ensure scarce and valued resources is evaluated, and
(c) a process approach where the behavior of organizational units is evaluated (Murphy et *al*., 2013).

In the following section we will describe the most widely used methods for measuring organizational performance: The Balance Scorecard method, Structural Equation Modelling, Performance Prism, Performance Pyramid and Triple Bottom Line Approach. All the methods are very useful and precise tools for measuring the reflection of organizational knowledge in organization performance.

The Balanced Scorecard (BSC) is used for strategic planning and management when companies or organizations want to put on paper what is the goal they want to accomplish. BSC is also used to align the everyday usual work with companies' strategy. Moreover, it is used to measure and observe the progress towards to strategic targets (Malina, M. A., & Selto, F. H., 2001). This method is used by companies in various fields. BSC is used e.g. in companies and organizations such as Volkswagen, Ford Motors, Philips, UPS, Apple, FBI or University of Virginia.

The original idea of the balanced scorecard comes from Dr. Robert Kaplan and Dr David Norton and was created to measure organizational performance with the help of more balanced set of performance measures. It added the nonfinancial strategic measures to the traditional financially oriented measures. The BSC system has developed over the years and is considered now as a fully integrated strategic management system (Kaplan, R. S., Davenport, T. H., Robert, N. P. D. K. S., Kaplan, R. S., & Norton, D. P., 2001).

The BSC is a link that connects the operational goals and activities with the strategical goals and concepts as mission, vision, and core values. It is considered as multidimensional since it contains both financial and non-financial indicators. The system includes four perspectives: financial, customer, internal processes and innovation and learning perspective.

The financial perspective contains indicators as economic value added, revenue growth, costs, profit margins, cash flows and operating income. It monitors the financial performance and the use of financial resources (Kaplan, R. S., & Norton, D. P.,1996).

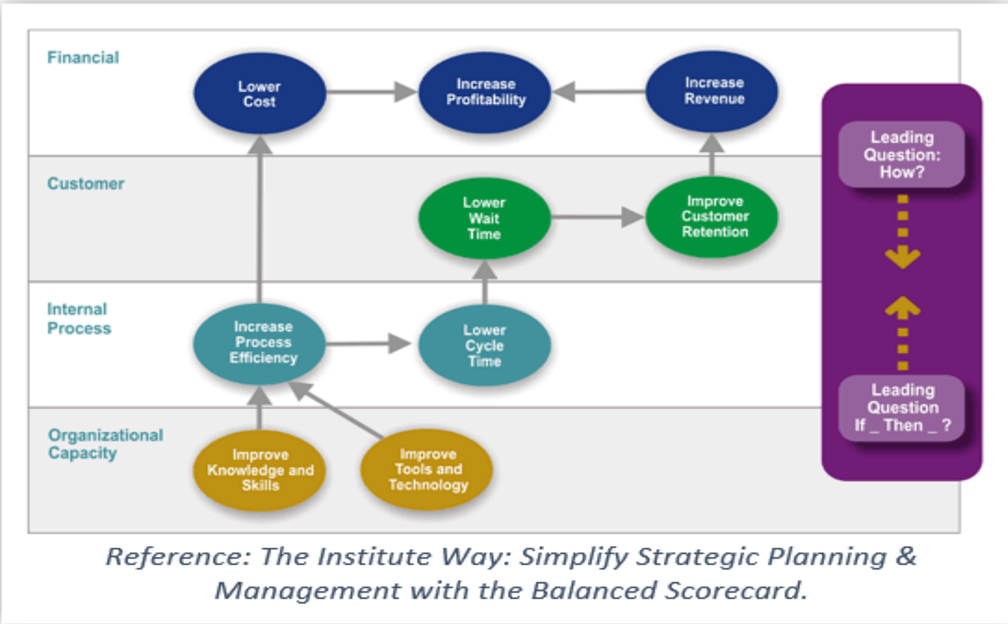
The customer perspective is oriented on the value suggestion that a company will apply to satisfy their customers and generate more sales to the most desired customer groups (Alejandro, et al., 2011). On one hand it includes the value for the costumer as quality, performance, service, or time. On the other hand, there are the indicators as customer satisfaction and market share.

The internal process perspective covers all the activities and processes necessary for the company to succeed at providing what is expected from the customers (Figge, F., et al., 2002). It overviews the efficiency and the quality connected to product or services. The operations management, customer management, innovation and regulatory and social issues belong here.

The fourth perspective: innovation and learning perspective or innovation and growth - is oriented on the intangible assets such as internal skills and capabilities required to support the internal process creating values. It monitors the human capital, infrastructure, technology, culture, etc.

Some researchers (Zeynep Tuğçe Kalendera, Özalp Vayvay, 2016) include also a fifth perspective and that is the supplier perspective which can also play an important role in assessment of the non-financial performance.

For each of these 4 (or 5) perspectives the BSC recommends developing objectives, key performance indicators (KPIs), targets and actions. The drawback of the Balanced Scorecard is the fact that it ignores the role of different stakeholders that they have by the success or failure of a company.



Picture 2. BSC Measures - Key Performance Indicators, (Source: Kelly, 2015)

By creating the strategy map, it is necessary to include at least one Key performance indicator (KPI) to be monitored and measured over time. These KPIs should specify the progress toward the outlined goals and also show the gap between the targeted and actual performance.

As the KPIs are well structured they should provide an objective way at the functionality of the strategy, to focus attention on what helps to be successful.

In a big company cascading can be used - that means creating of various tiers for different levels of the company. The first Tier is used as a wide corporate scorecard, then Tier 2 for business and support units and Tier 3 for teams or individuals. Cascading should interlink the work people and the management to team up for the same strategical goal. As the strategy cascades down through the company, the goals, and objectives as well as the performance measures become more operational and tactical (Ahuja, S., 2012).

After developing and implementing the BSC it is recommended to use performance management software to transfer the right performance information to the right people at the right time. The QuickScore Performance Information SystemTM is one suggestion.

The balance scorecard can incorporate four parts customer, learning & growth, internal processes, and finance. These should be achieved through stating the objectives, measures, initiatives, and actions. Objectives are goals on an organizational level. They should be interlinked to strategical goals. Measures should be included in order to find out and understand how successful is achieving of the objectives. It is suggested to have 1-2 measures per objective. In order to achieve your objectives BSC includes initiatives as key action programs. Out of the BSC they can be called projects, actions, activities. For each objective, an organization usually has 0-2 initiatives. Tasks that are according to BSC delegated to an employee or a team are called actions. They are not necessarily a part of BSC framework, but they help to achieve the key initiatives and are included in the processes (www.clearpointstrategy.com).

The best management tool to look forward and backwards and to show how your today's action will improve the performance in the future is the Balance Scorecard method.

Structural Equation Modelling (SEM) modeling is a widely used statistical technique for behavioral science. It represents a combination of factor analysis and regression or path analysis. Its roots arise from path analysis invented by Sewall Wright, geneticist (Wright, 1921). A path diagram is still usually used as a start to SEM. In the original, the observed or measured variables are represented by a rectangle or square box and the latent or unmeasured factors by a circle or ellipse. Single headed arrows represent causal relationship – variable at the tail causing the variable at the point. Double headed arrows indicate covariance or correlations without a casual representation. From the statistical point of view, single headed arrows represent regression coefficients while double headed the covariances.

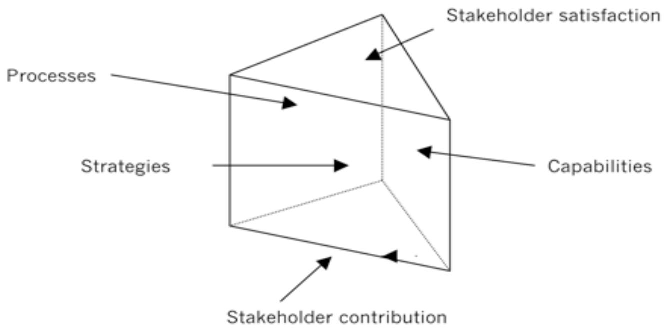
The SEM includes the factor analysis and regression but at the same time offers more flexibility by combining these two techniques together. It is well equipped for causal analysis or cases with multicollinearity. The first part of SEM is called the measurement model and it is similar to factor analysis. The second part is called the structural model and this part relates the parts of the measurement model with dependent variables. Sometimes the measurement model is skipped then path analysis is a better name then SEM.

SEM can be used for various types of data such as economic data, social media data, survey data or MRI data in any data type – nominal, count, interval or ratio. With SEM company is well equipped for customers perceptions associated with liking, purchase interest, customer satisfaction of your products (Sharma, G., & Baoku, L., 2013).

To be able to construct SEM we need a minimum of 200 cases or respondents are needed and at the same time at least 10 cases per measured variable. That means minimum of 250 respondents if there are 25 attribute ratings in the model (Lowry, P. B., & Gaskin, J., 2014).

If you are not sure if your model is good you can use so called fit indices. The most common fit indices Comparative Fit Index (CFI) and Root Mean Square Error of Approximation (RMSEA) also the well-known R squared is helpful.

Performance prism is a measurement system for management that answers five main questions. Its goal is to meet the requirements of all stakeholders. This model is used by organization such as DHL. The approach of the performance pyramid is to start from the stakeholders' requirements and from these requirements start to develop the performance measures (Neely, A. D., Adams, C., & Kennerley, M., 2002).



Picture 3. The performance prism, (Source: Neely, A. D., Adams, C., & Kennerley, M., 2002).

The performance prism belongs to the second generation of performance management frameworks. Its substance is formed out of 5 interrelated facets whereas the importance is given to gathering of information rather than to simple measurements. Performance prism is a flexible tool that reflects the complexity of performance measurements and management.

According to Neely et all (2001) the five facets are:
Stakeholders satisfaction which defines who the stakeholders of the given company are and what it is that they actually want to do. This first part consists of stakeholders' mapping. This means the description or mapping of who are the stakeholders, how important are the individual groups of stakeholders, what power they possess and what their readiness to use this power is. Since one of the main goals of the company should be to keep the important groups of stakeholders happy in this part of performance prism the company should find the performance measurements through which it will be able to measure this happiness. The main groups of stakeholders usually are investors, customers, employees, suppliers and joint ventures partners, and regulators.
Stakeholders' contribution – this facet shows the contrary and that is what we want from our stakeholders. Besides knowing who the individual groups of stakeholders are and what they expect from the company it becomes more and more important to define the expectations that companies have from the individual groups of stakeholders. When we find out what the expectations are then it is necessary for us to find the appropriate measures to measure the level

of providing these expectations. For example for customers - as one of the stakeholders groups - the costumer profitability can be such measure.

Strategies should include the types of strategies we should implement in order to satisfy the needs and wants of our stakeholders by achieving our goals. First of all, a clear distinction between strategy and long-term goals should be made. In the performance prism the importance is given to strategy – so the way how the goals will be achieved. In this part we don't define the goals - they are defined in the first two facets. Here we are looking for the way how to reach them. So, we are looking at performance measures that check if the strategies that the company chooses in order to achieve its goals ensure that the goals and needs of the stakeholders are satisfied at the same time. The performance measures here should show how well the strategies are being implemented. That means the management should find the best ways how to communicate and spread the strategies throughout the whole organization, how to encourage the willingness of the managers on various levels to implement these strategies. The important work consists also from monitoring if the strategies are still up to date.

Processes describes the processes are necessary to complete in order to implement our strategy. In this part it is crucial to monitor if for the appropriate strategies the company is implementing the appropriate processes to support the strategies. Four main processes are usually defined

a) Development of products and services.
b) Creation of demand.
c) Satisfying demand.
d) Planning, organizing and managing the company.

These are the main processes that should be divided into sub processes and each sub process and process should be given an owner. In the next step, the company has to develop the measures through which it will control the functioning of the processes. The process reengineering as well as value chain analysis can be used in this facet.

Capabilities facet comprises the capabilities needed in order to operate the processes from the fourth facet. To capabilities belong the people, practices, technologies and infrastructure necessary for the processes to work. In this facet the company identifies the needed capabilities and performance measures to identify how the capabilities are being performed.

The individual facets of the performance prism should be interlinked in a top to bottom process very similar to performance pyramid. The difference here is the orientation on more groups of stakeholders, on their needs and also on the expectations that the company has from these groups of stakeholders (www.accaglobal.com).

1 Stakeholders' satisfaction

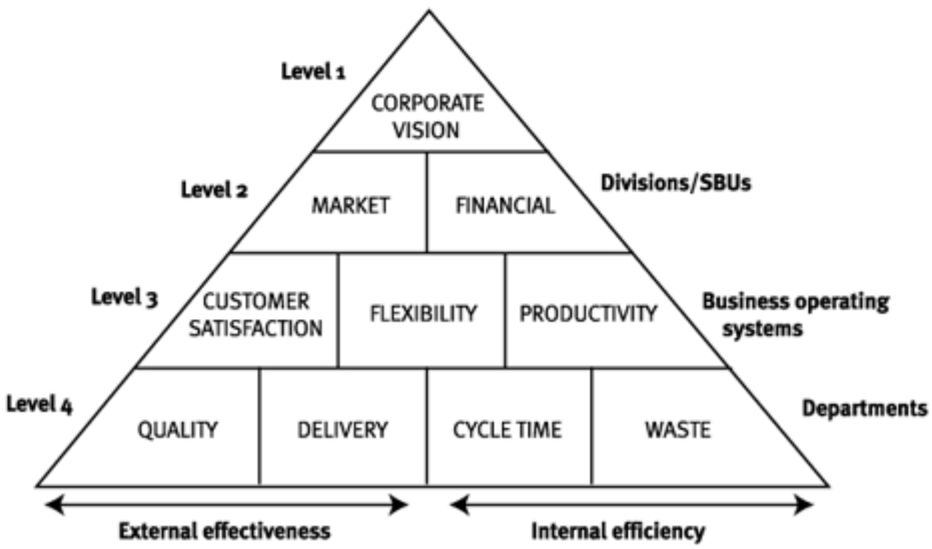
2 Stakeholders contribution

3 Strategies

4 Processes

5 Capabilities

Performance pyramid belongs to the systems of performance evaluation that incorporate both financial and nonfinancial aspects (Kaplan, 2012). Performance pyramid was developed as a hierarchy of financial and nonfinancial indicators. It attempts to show the link between long term strategic and short-term operational goals.



Picture 3. Performance pyramid, (Source: https://kfknowledgebank.kaplan.co.uk/the-performance-pyramid-)

The top level 1 describes the vision or mission of the organization. It is dedicated to firm's long-term success and competitive advantage. The Level 2 is oriented on critical success factors of the company measured with market related and financial measures. On the level 3 there is the marketing and financial success linked to achievement of customer satisfaction, higher flexibility and higher productivity. The accomplishment of the goals from Level 3 can be monitored through the indicators of Level 4 which are quality, delivery, cycle time and waste. Thy pyramid can be divided into two sides – on the left side there are mainly non-financial indicators with an external focus. On the right side there are financial measures oriented on internal issues.

The performance pyramid forms a pyramid of performance targets and measurements with the impact on linkages between operational performance and achievements of the strategic goals. Besides the financial goals the importance is given also to customer satisfaction, flexibility and productivity. The pyramid obviously distinguishes between measures important to external parties and measures important within the company. According to Lynch and Cross, it is important to make a connection between strategic and operational goals for a company. Especially, the fact that the individual departments know the extent to which they are contributing to the achievement of the goals is crucial. The used measures should include financial and non-financial information available. This information should be necessary when and where it is needed. The main goal of the performance pyramid should be to focus all business activities on the needs and requirement of their customer.

The disadvantage of the performance pyramid is that is does not take into account two groups of the stakeholders – shareholders and customers.

Triple bottom line approach (TBL) developed by John Elkington (Elkington, J., 1994) was a new approach to performance measuring, including not only the traditional measures such as profit, return on investment, etc. but also the social and environmental aspects of business. Thanks to the included aspects of people and planet it has become an important tool to support the sustainability goals and performance evaluation.

In the 90ties a new framework to measure performance of companies was established by John Elkington. He called this accounting framework the triple bottom line (TBL), tried to go further than just the obvious measures such as profit, return on investment and stakeholders' value. He wanted to include the environmental and social dimension with the idea of sustainability goals in his mind.

TBL is a framework with three dimensions of performance: financial, social and environmental. Compared to the traditional financial frameworks including the social and ecological dimension in TBL means increasing the difficulty of finding appropriate measures. Sometimes it is called also 3 Ps: people, planet, and profits. TBL is used by profit and non-profit organizations as well as all levels of governments. (Slaper, T. F., & Hall, T. J., 2011).
The name comes from three dimensions of performance: financial, social and environmental or also so called 3Ps: people, planet and profit. The difficulty does not come with the dimensions it comes with their measurement. One approach would be to monetarize all three aspects. This approach brings a common unit – dollar or euro – but sometimes it is very hard to monetarize endangered species. Another possibility would be to use an index such as Indiana Business

Research Center's Innovation Index. Another possibility is to measure each measure alone. So far, no universal measure exists. Therefore, the set of measures can be determined by the stakeholders according to the size, type of the project measured or the available data. As for the economic measure indicators such as income, expenditures, employment and taxes can be used. Environmental measures should measure the natural resources and potential influences on their viability. To this category belong natural resource, air, water quality, energy consumption, and use of land.

Social measures could include health and wellbeing, education, access to social resources. On the company's level the main challenge is to develop a comprehensive and meaningful index and to identify the appropriate data for the index composition.

With profits measured in dollars or euro or any other currency, it is not so easy to find measured for social capital or environmental health. Some theories say the way to go is to monetize all three measures – which gives us a common measure. But it is not so easy to calculate the damage on environment or endangered species. Another possibility would be to calculate an index if there was a consensus on comparing the performance between various companies, cities, etc. Another possibility would be to have an individual measure for each variable. That's why there is no universal measure how to calculate TBL so far. There is the chance for the stakeholders to determine the set of measures depending on data availability (Andrew Savitz, 2006).

To the economic measures one can include profits, income or expenditures, employment, percentage of firms in each sector, revenue of the sector, taxes. The environmental measures could include air and water quality, energy consumption, solid and toxic waste, land use etc. And the examples of social measures could incorporate unemployment rate, access to social resources, quality of life, average commute time etc.

The TBL is used by businesses, nonprofit organizations as well as by governments. All of these entities are motivated by the principles of economic, ecological and social sustainability even though they use different ways to measures the outcomes.

For business using TBL offers the possibility to foresee the long-term sustainability of the company. As for the measures – economic - usually taxes paid, social – average training hours per employee, charitable contributions, environmental - safety incident rate, greenhouse gas emissions, water consumption.

The concept of TBL offers organizations large flexibility and foundation for measuring the long-term sustainability. The drawback is the measuring and finding the appropriate measures and available data.

3 Conclusion

We have shown the importance of knowledge to performance of an organization. In the first part we have looked at the psychological aspects of knowledge acquisition, retrieval use and possible application to knowledge measurement. We have also shown how the knowledge management SECI model can contribute to understanding of the knowledge flow in an organization. In the next part, we have shown a more practical approach to measuring knowledge in an organization. We called it a beehive approach in which we do not focus on the knowledge possessed by an individual but at the knowledge on an organizational level that becomes visible in the organizational performance.

Consequently, we have described several methods for measuring the organization performance as a result of a collective knowledge in an organization. Each approach has its positives and drawbacks. Our aim was to show that knowledge measurement is possible and is more exact on an organizational level rather than on an individual level.

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VIII

A method of measuring the added value of facility management as a competitive advantage and a knowledge generating tool

Christian Schlicht

1 Introduction

"Measure what is measurable and make measurable what is not so." - Galileo Galilei

The VUCA world, as acronym for volatility, uncertainty, complexity and ambiguity, forces companies to face divergent market challenges and to change old strategies into new ways of thinking and solutions (Abidi, Manoj 2015). Facility Management (FM) companies and corporates with real estate management departments need to adjust to increasing digitization, increasing sustainability requirements, changes on the tenant market, and the increasing competition for talented employees. To keep up with competitors and dynamic requirements of stakeholders, innovations have become a crucial success factor in the Facility Management market.

The management of ECE Projektmanagement G.m.b.H & Co. KG (ECE), the German market leader in the shopping center industry, for example, faces high transparency by investors going along with flexible, innovative and customer focused operation models. Whereas tenants and retail partners ask for professional management, benchmarking, marketing, as well as a variety of center events (ECE Market Report 2015). Additionally, it is important to stay attractive for visitors in the variety of retail partners and in the gastronomic tenant mix and to create customer experience and convenience in the malls. To meet the customers' needs, it is therefore important to have a "comprehensive management approach" which involves leasing, marketing, operational center and Facility Management as well as the commercial management of the technical assets and the real estate itself (ECE Market Report 2015). It is therefore crucial to steer Facility Management processes towards precise targets and to measure achieved goals and highlight added values to the complex product of a shopping center. Due to the different changes in the market and the impact on the involved parties, Facility Management needs to become a more strategic focal point in the future – as a management discipline that touches all the stakeholders and constitutes the interconnection in the value chain of real estates. The paper aims, therefore, to measure the added value of the Facility Management department within a real estate management supply chain using a case study of ECE regarding the management of shopping centers.

2 Measuring added value in Facility Management

2.1 The idea of Added Value in FM

In the meantime, different models for the description of an Added Value are available in the literature (Jensen et al., 2013, Jensen, 2012, Lindholm and Leväinen, 2006) and the nature of the benefits of a Facility Management (FM)[17] department is widely described in the literature and has become increasingly important in recent years (Jensen, 2010, Jensen et al., 2013). The reason for this is that FM has so far been regarded only as a "Cost Collector", although the management discipline contributes significantly to core business' value creation and to the entrepreneurial competitive position by controlling secondary processes (Jensen et al., 2013). However, this contribution is not clear to many managing directors and companies. It is, therefore, important to highlight those benefits in quality and quantitative measures. And so this paper don´t aims to discuss the different perspectives or the different ways in which Facility Management can create added value to the core business; the aim is to develop an instrument, which could combine with other structures and elements, like the FM-Value Map (Jensen, 2012).

One of the main problems is that it is unclear how to measure the benefits that come with professional and effective Facility Management. The measurement of the benefits of a FM department has not yet been operationalized and has been mostly conceptual in nature. It is essential that the FM targets are derived structurally from the corporate strategy and that the added value is clearly related to the core business.

[17] FM is understood as the "integration of processes within one organization for the provision and development of the agreed services, which help to support and improve the effectiveness of main activities of the organization." DIN 2006. DIN EN 15221: Facility Management-Begriffe, Deutsche Fassung. Berlin: Deutsches Institut für Normung e.V.

Initially added value can be described as a “trade-off between benefits and sacrifices in connection to organizational objectives (added value)” (van der Vordt et al., 2016). But for the measurement of the theoretical construct of added value it is crucial to identify all relevant strings and factors which lead to a specific added value.

Unfortunately, this idea of assessing added value is not applicable for FM even though there is a clear focus on human factors such as building users, too.

2.2 Research approach and aggregated results

As a result of the complex phenomenon of defining and measuring the added value of Facility Management a mixed method research approach was used, consisting of three different methods with coherent research questions:

At first, a structured literature review based on BECKER (Becker) addresses the research question, if concepts, models and/or methods are available in Corporate Real Estate Management (CREM) and FM and other domains, which are suitable to measure the added value or the impact of an action. According to this approach the research focus was derived as illustrated in the following table.

Attribute	**Characteristic**			
Focus	Results	Methods	Theories	Application
Target/Aim	Criticize	Integration (Generalization)		Challenges
Perspective	Taking an position		Neutral representation	
Coverage	Complete	Completely selective	Representative	Central
Organization	Historic	Conceptual		methodical
Target group	Professionals	Researcher	Practitioner	Public

Table 1: Characteristics of the literature review; colored background = research focus

It showed that there is still no common definition for added value or rather different perspectives, neither within the different sectors considered in the literature analysis (sustainability, hospitals, logistics, etc.), nor in FM; this aspect was also listed by JENSEN et al. in 2012 (Jensen et al., 2012). Even more, most tools are measuring the performance of Corporate Real Estate Management (CREM) / FM (output) in a functional perspective without the linkage to the original organizational performance (van der Vordt et al., 2016). Also, the literature review showed that many added value approaches in FM are limited to conceptual models which barely take key figures, key performance indicators (KPIs) or benchmarks into consideration. The practical use of those concepts and the assessment of achieved added value is therefore limited and hardly to handle in business context.

Within the literature analysis more than 65 sources brought up 72 different added value models/approaches from various industries. These models were classified and clustered using the following criteria, illustrated in Table 2.

Table 2: Model rating criteria in context of the literature review

Criteria	Characteristic	Model Rating
Operationali-zability	In what form does the model already integrate key figures?	from 1 = no key figures and specific context for measuring the added value in an organizational perspective to 6 = integrated key figures and linkage to the key figures on the organizational level
Efficiency	In what form does the model integrate measures at an organizational level?	from 1 = description only at the level of cause and effect to 6 = models with a focus on measuring the "added" value in an outcome orientated perspective

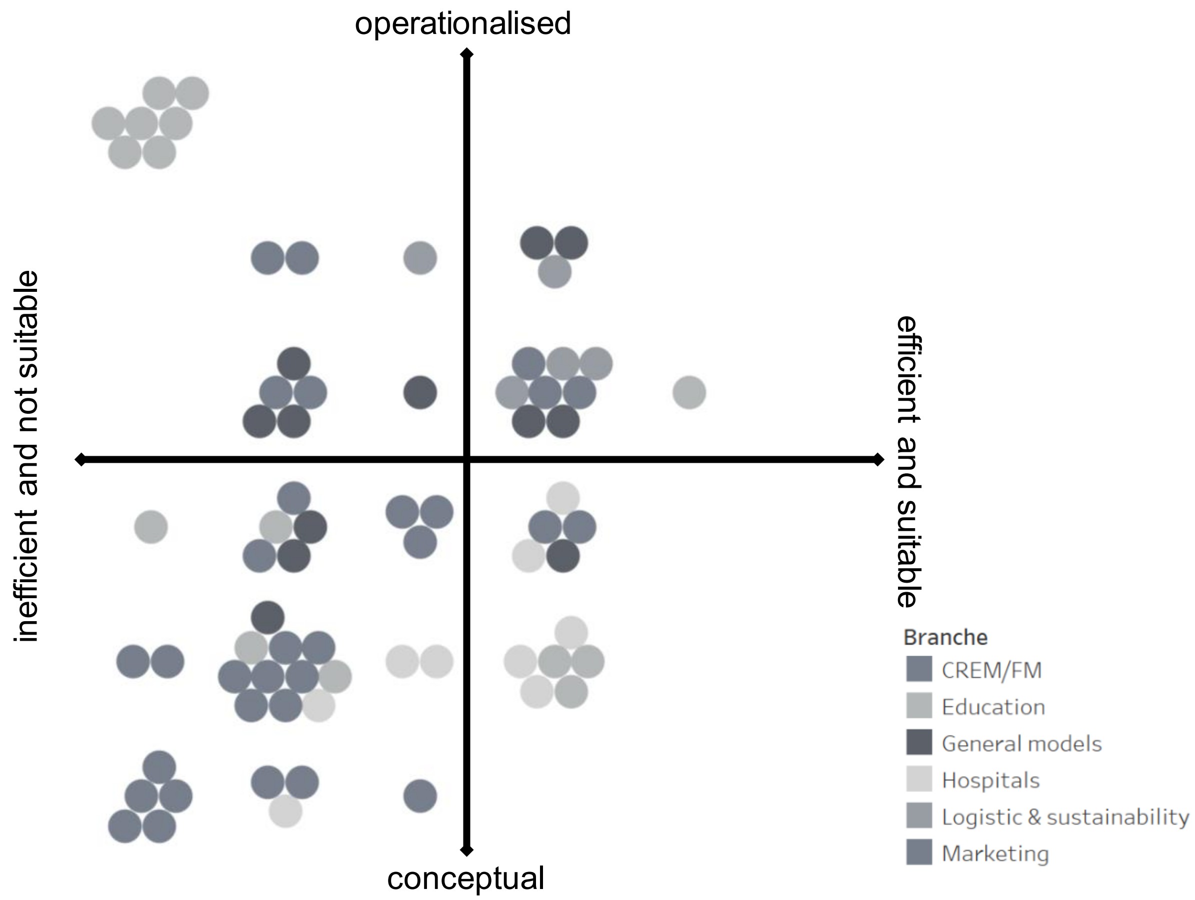

Figure 1: Result of the model clustering

One idea of added value is its interpretation as a pure advantage in the manner of the sum of different outputs. Doing so in a business context every achievement of a business department or operational process goal can be seen as an added value to the company goal. This definition is the base of many models which solely focus on goal fulfillment. Another idea of added value is a broader view of goal fulfillment, that means it is not only about meeting targets, but over-fulfillment and outperformance of processes, business ideas or departments goals (European Commission, 2011). The EU pointed out, that on "a general level, […] added value is the value resulting from an EU intervention which is additional to the value that would have been otherwise created by Member State action alone." (European Commission, 2011).

As a second part of research, a quantitative study - based on the theory of planned behavior (Ajzen and Madden, 1986) – had the mission to describe the relationship between attitudes and behavior with regard to the measurement of added value of Facility Management. The model from AJZEN and MADEN is particularly determined by a concrete behavioral intention, which itself is influenced by three variables, the attitude, the subjective norm and the perceived control (Ajzen, 1991, Kanning et al., 2008). The investigation was carried out using a questionnaire. Following this definition were 23 questions on the variables of the theory of planned behavior. The answer options are orientated on a six-point scale (e.g. from 1 = fully agree to 6 = completely disagree) and are operationalized from the following (aggregate) hypotheses that were formulated by the author:

Table 3: Use hypotheses on a general level

Dimension (Factor)	No.	(aggregate) hypotheses
Attitude	H1	The more positive the employees' own attitude towards measuring added value in CREM/FM, the stronger their intention to measure it (act on it)
Subjective norm	H2	The greater the perceived social pressure to measure added value, the stronger the behavioral intention to measure it (to act)
Perceived control	H3	The higher the perceived control, the stronger the behavioral intention to measure it (to act)
Perceived control	H4	The higher the perceived control, the more likely the behavior to implement a measurement of added value
Behavioral intention	H5	The stronger the behavioral intention to measure added value, the more likely it is to be implemented

A total of 30 full responds were integrated in the further data analysis that was coordinated by the author; the relevance of the included 23 items was verified through a factor analysis and the exclusion of 4 items. The overall results in context of the underlying theory are shown in Figure 2.

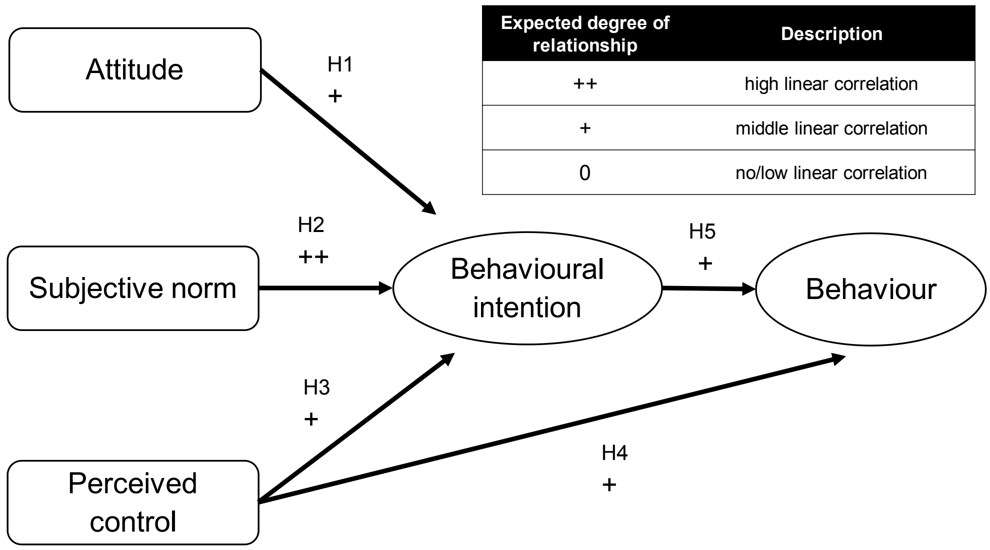

Figure 2: Results for measuring the added value (Bernhold et al., 2018)

The third used approach in the methodological research set was the case study approach. The core task of the case study (Yin, 2003, Mayring, 2002, Woodside and Wilson, 2003) was to develop a measuring approach including a prototype. On a general level, added value can be seen as any overfulfillment of the objective set by the company (i.e. the overfulfillment of objectives which should have been achieved even without special measures). The case study research involves a single case approach with embedded units at ECE; ECE is a shopping center specialist and one of the biggest asset managers in Germany with 33 Billion EUR assets under management. It was assumed that the added value of the CREM/FM would be generated by the involvement of different company functions. The added value is thus to be regarded as a key indicator of target achievement (or overachievement) and thus a cross-functional result. The embedded units are shaped through the Asset, Leasing and Center Management as well as through the Facility Management of a center/shopping mall.

The advantage of this case study approach lies in the acquisition of intensive and strategic insights into the development of objectives on the one hand and the comparison of objectives between functional areas on the other. The problem-centered interview was used as the data collection method and all interviews were recorded (Mayring, 2002; Hussy, 2013). The catalogue of key questions was divided into different categories:

- General understanding of the term “added value”
- Objectives / defines (e.g. real estate strategy, objectives)
- Integrated stakeholders / shareholders
- Added value (e.g. indicators, percentage equity share of result, possibility of influencing)
- Sanctions (expected sanctions in case of target failure)

The recorded interviews were prepared in summary protocols; the individual statements were assigned to the individual aforementioned categories for evaluation purposes. According to the communicative validation (Mayring, 2002), the prepared protocols were made available to the interview partners again. One of the main case study results were the different KPI´s on the task level of a shopping center, which act as valuation surrogates in the measuring process.

2.2 Conceptualization of an Added Value Measuring tool for CREM/FM

In general, the added value is a complex hypothetic construct (Homburg and Giering, 1996) which cannot be observed directly. In this context, the conceptualization will include the development of constructional dimensions and the development of a measurement instruments as a model operationalization (Homburg and Giering, 1996). From this perspective, the added value at the strategic level can be multi-sighted and multi-dimensional; LINDHOLM and LEVÄINEN (Lindholm and Leväinen, 2006) for example offer an initial starting point at the real estate strategy level.

Based on a descriptive analysis of the essential and relevant models from the literature analysis, the Balanced Scorecard (Kaplan and Norton, 1992, Kaplan and Norton, 2001), the Tableau de Bord (Bourguignona et al., 2004, Daum, 2005) and the FM Value Map (Jensen, 2012, Jensen and Katchchamart, 2012, van der Vordt et al., 2016) can be understood as adaptable models. The Tableau de Bord is particularly suitable as a model, since it combines the strategic perspective with the operative implementation and is also able to map a hierarchical structure. The Balanced Scorecard is very suitable for the mapping and visualization on an aggregated corporate level and the FM value map is especially suitable for the systematization of measures and the measurement of the results. One topic of the case study was to develop a tool, which is able to combine the strategic corporate level with the operational level and which is suitable for differentiating the degree of goal attainment in a cross-functional (horizontal between functions at the same corporate level) and hierarchical perspective (vertical within one function). The Tableau de Bord is a very practical and valuable instrument, because it gives the user a quick insight into actions and their status at different level (Bourguignona et al., 2004). Due to the lack of predefined categories - as given at the BSC - a consistent derivation of the CREM/FM strategy from the corporate strategy appears essential.

With consideration of the multidimensionality, the model was conceptualized through the following level:

Table 4: Hierarchy model levels

Level	Main topics and description
Corporate Level	Description and derivation of the overarching corporate objectives (e.g. customer satisfaction) [the possibility of using the target elements, e.g. of (Lindholm and Leväinen, 2006; (Jensen and van der Voordt, 2016)]
Functional Level	Description of the share of each corporate function (e.g. CREM/FM or Logistics) in this goal; in total the result over all corporate functions must be 100% result
CREM-FM-Level	Description of the share of each CREM/FM department (e.g. Asset Management or Building and management) in this goal; in total the result over all CREM/FM-functions must be 100% result
Task Level (Indicators)	Definition of the KPIs used to determine whether or not the objectives have been met

The framework assumes - in accordance to the Tableau de Bord - a cascaded development in a top-down manner and begins therefore with the corporate level. From this point, the KPIs at the task level are only an operationalized reflection from the company strategy on the top level and furthermore, the KPI is more like a surrogate, which combines the observable real world with the conceptual added value model.

3 Case study results and Operationalization

After conceptualization, one of the main tasks of the case study was the operationalization of the model and its practical application. The indicators at the Task Level (see table 5; result of the case study) act as Key Performance Indicators (KPI). Each KPI is weighted with regard to their meaning from 1 to 10 (1 = very low importance; 10 = very high importance) and the weighting is carried out in accordance to the Analytic Hierarchy Process (Saaty, 1986). Every KPI has a target and an actual level; table 5 give an overview over the calculation process as an example of the Asset Management Department of the case study.

Table 5: Measuring of goal achievement with track record

No.	Indicator	Unit	Target value	Actual value	Track Record (goal)	relative weigh-ting	Weigh-ting within the depart-ment	Track Record (weigh-ted)	Rating
1	Return	%	3	4	1,333	8	25,00%	0,333	Outperformer
2	Amount of rent	EUR/ sqm	6	6	1,000	4	12,50%	0,125	goal achieved
3	Ancillary costs	%	2	2,3	1,150	8	25,00%	0,288	Outperformer
4	Influenceability of ancillary costs	%	50	49	0,980	2	6,25%	0,061	Underperformer
5	Reliability of budget planning	%	95	100	1,053	5	15,63%	0,164	Outperformer
6	Real Estate value	EUR	100000	90000	0,900	5	15,63%	0,141	Underperformer
n									
In total	/		/	/	/		**100,00%**	**1,112**	/

The interdependencies of the numerous identified KPIs related to the management of shopping centers are highly complex. Therefore, the track record principle was included to assort the indicators by paying in on destined targets. The track record is a scale of success for the single department as part of the CREM/FM function and is, at first, a conceptual measurement model that covers the relations between the different stakeholder groups, the outcome as well as the taken measures.

Table 6: General characteristic of the Track Record

Value of Track Record	**Degree of success**
TR > 1	The set targets are exceeded (Outperformer)
TR = 1	The set targets are achieved (Goal achiever)
TR < 1	The set targets are not achieved (Underperformer)

From a mathematical perspective, the TR can be described as follows:

$$TR_{jk} = B_k \times \frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i} \times \frac{Output_j}{Input_j}$$

Equation 1: Measuring the Added Value

with:

$Output_j$ = actual value of a KPI_j

$Input_j$ = target value of a KPI_j

α_j = relative weighting of a KPI_j (in accordance to the other weightings)

B_k = endogenous influence factor of CREM/FM-department k

n = Total number of KPIs considered

TR_{jk} = weighted track record for KPI j of CREM/FM-department k

Using the model, it should be possible to measure how much one factor affects the other and what impact this has on the return - for example, how customer satisfaction contributes to the level of return through other factors. The weighted values within the model should make it possible to measure the actual given share that FM contributes to the return.

Bearing in mind, that the Tableau de Bord is a suitable decision support tool to assess and operationalize added value in an efficient way, we complement this model with the idea to

- create a holistic overview
- ensure a cause-and-effect understanding,
- translate a company vision and strategic objectives to operational company levels in a top-down manner,
- create a generic overview of added values at an aggregation level that enhances fast steering and decision adjustments.

Resulting from this, the Track Record Model for measuring the added value of Facility Management was developed.

The model uses the definition of added value whereas added value is given when defined targets are overachieved or goals are over met. This is assessed by the model in stating the measurement units as so-called outperformance. Meeting directed goals on point is stated as goal keeping, whilst not achieving a goal is compared to underperformance and a “negative” value.

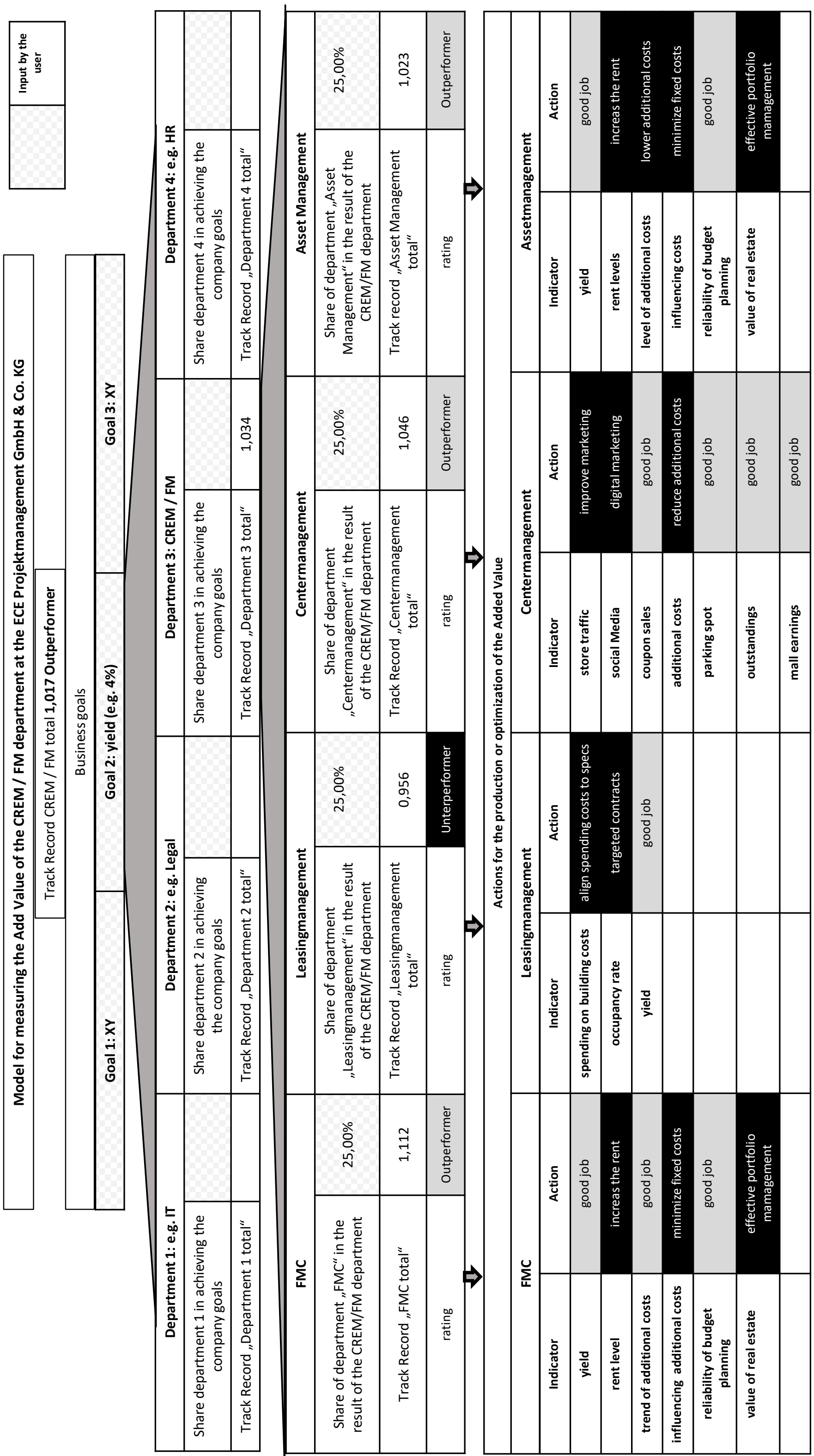

Figure 3: Example of the track record model for measuring the added value of FM at ECE

The model focuses on specific department goals that were derived directly from the corporate strategy as shown in figure 6. This means that every department has its own value creation target to contribute to the company's overall goal. Targets related to the KPIs may be adjusted dynamically by the company. Resulting from the combination of input, output and weighted assessment factors of the KPIs, the track record for each department and the associated targets is calculated. The sum of the track records of FM related departments (Center Management, Asset Management and Leasing Management as well as Facility Management) results as an entire track record of the overall FM efforts.

The results of the calculation are displayed in the model based on three categories. If the track record calculation for a target results is a value smaller than 1, the target for the corresponding indicator has not been reached, and thus the department is classified as "underperformer". If the track record result is exactly 1, the respective target was reached on point; the result is classified as "goal achieved". Whilst, track record values higher than 1, are highlighted as "outperformer".

Of course, categories for track record values may be modified for different business cases easily. As the track record model for added value is a comprehensive model for the use in different departments and various business units it supports a management display using the simplification of a KPI dashboard.

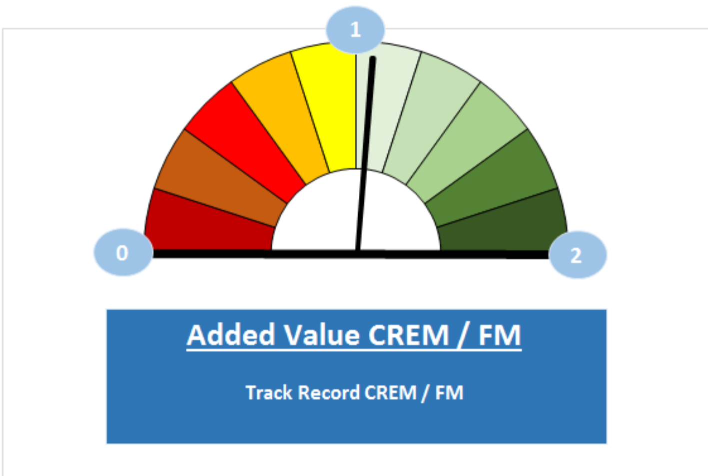

Figure 4: Dashboard of the track record model

4 Double Touchpoint Pilot Project: FM is Making the Customer Journey a tangible Experience

As traditional retail concept, how to compete in a constantly growing e-commerce and online shopping environment? The answer is easily at hand: by delivering visitors an excellent customer journey. ECE with its claim as European market leader in the shopping center segment analyzed the individual touchpoints forming part of a comfortable customer journey in the course of the *At Your Service-Study*. (ECE - At your Service 2016) With a clear focus on the customer, six touchpoints have been identified as core aspects contributing to a positive shopping experience: home, arrival, information, shopping, relaxe and departure. The detailed design of these touchpoints in accordance with individual customer needs is essential for a successful customer bonding and thus forms part of the core DNA of all shopping centers in times of the digital change. (Bringmann, 2013)

An integrated FM approach can contribute to an increased customer satisfaction. When it comes to touchpoints, the excellence lies in detail. Targeted FM measures can thus make the difference. With the aim of creating an added value for the shopping experience, ECE conducts customer surveys on a regular basis. In this respect, one customer touchpoint in a double sense instance has been identified as clearly improvable: escalator handrails. Visitors indicated to consciously not touch handrails as the surface is assumed to be unsanitary and contaminated with bacteria. From a FM operator's perspective, however, handrails represent an important factor when it comes to customer safety. A contradiction which urgently had to be solved by a smart FM innovation. In cooperation with UV-Innovative Solutions GmbH (UVIS), ECE consequently started a pilot project for increasing visitors' shopping comfort in terms of handrail sanitation.

The pilot project has been conducted in one ECE shopping center (Phoenix Center Harburg): four escalators have been outfitted with the UVIS self-disinfecting handrail technology which eliminates 99.99% of all germs located on the handrail surface. (UVIS, 2019) The measurement of the actually experienced added value induced by the installation has been realized by the conduction of a representative, empirical survey. With a sample size of 637 customer respondents, a zero measurement has been conducted before a follow-up measurement after the actual implementation of the UVIS technology took place. The measurement survey has been performed as a computer assisted personal interview (padCAPI). The survey period was from 28.09.2018 to 27.10.2018. Next to the general query regarding respondents' demographical aspects (e.g. age, gender, level of education), the main part focuses upon the frequency of individual shopping center visits, the personal satisfaction with the shopping experience as well as in the second step improvements noticed in the center's service & residence quality. The factors security, cleanliness and general center atmosphere have particularly been stressed in the interviews. If not mentioned proactively during the follow-up interview, respondent's attention towards the hygienic handrail improvements has been called intendedly by the survey team. Individual effects of the self-disinfecting handrails on the subjective perception have therewith been identified directly.

What is the value of any FM measure if the actual effect on the customers' experience remains invisible? A close look on empirical survey results is therefore imperative. As a core result, an increase from 68% to 81% in the overall customer satisfaction with the center visit has been observed.

Figure 5: Overall satisfaction with the shopping center visit

The outfitting of handrails with the UVIS technology lead to a significant growth of 19% in the customers' perception of the overall cleanliness factor. Also the visitors' security notion perceptibly increased by 7%. The handrail disinfection therefore positively influences the customers' perception of center security, cleanliness and atmosphere.

Figure 6: Assessment of individual aspects

One survey result of special interest is the changed notion on the overall shopping center image. Results as presented in figure 7 clearly demonstrate that the improved sense for one particular aspect *("Image of the shopping center is clean")* positively impacts the evaluation of arbitrarily connected touchpoints.

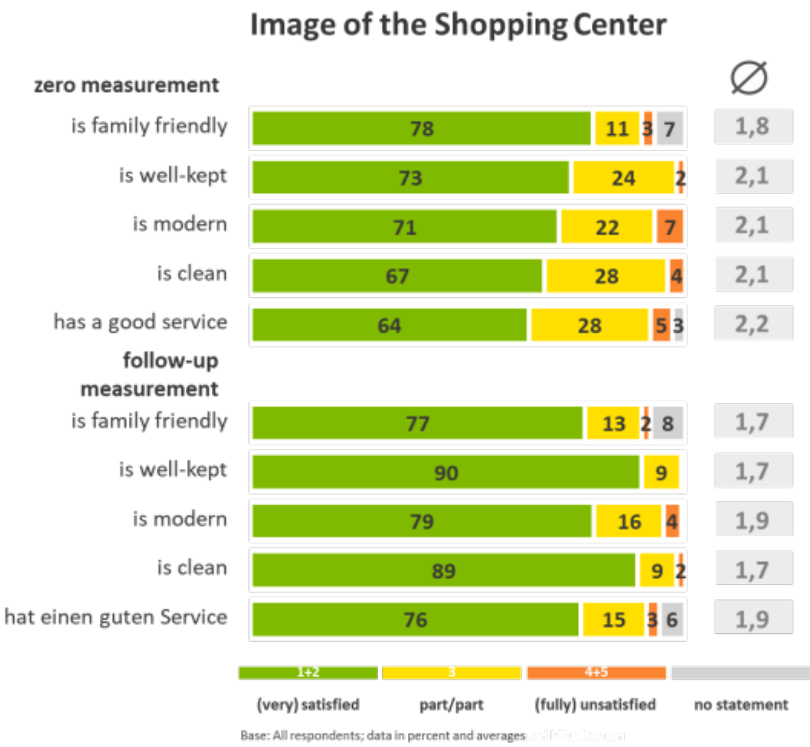

Figure 7: Image of the shopping center

The installation of the UVIS technology has been promoted prominently via embroiderers set-up shortly before the follow-up measurement. The survey, however, has demonstrated clearly that only 39% of the respondents have paid attention to that promotional activity. The key message of sterility has only been perceived by 23% of customers surveyed. The pure application of stickers is therefore not a sufficient mean of communicating the positive effect of disinfected handrails. Additional channels of communication therefore have to be identified in order to circulate the added value induced by the UVIS technology properly, especially among the – according to response quotes – underrepresented peer group of people with lower education and of higher age.

Quod erat demonstrandum: excellence lies in detail – minor FM innovations regarding touchpoint improvements thus can make the big difference it takes to succeed in creating a satisfactory and convincing customer journey. The reciprocal effect of a positive notional change regarding one touchpoint on the perception of other touchpoints is hereby distinctly to be emphasized. Especially the positive increase in the respondents' security notion (+7%) significantly underlines the spurring impact of small FM-related alterations fostering the entire customer experience by additionally improving none-related resilience factors simultaneously.

ECE successfully managed the down-streamed satisfaction of a customer need by the application of the Plan-Do-Check-Act methodology (Schneider et al, 2008): a touchpoint with an improvable demand has been recognized, a solution has been developed by consulting professional experts in the particular area of interest and a first testing of the installation has been conducted and measured. Based on pilot findings, the measure is currently being re-adjusted in terms of the added value and will – after another testing cycle in additional pilot centers – be rolled-out to the entire center portfolio afterwards. The measurable positive effect induced by handrail-disinfection on the comprehensive customer experience stresses the importance of FM in the customer journey improvement process.

The example of the *tangible shopping experience* clearly demonstrates that a customer journey in terms of operational & service excellence is closely related to modern and integrated FM activities. FM therefore takes over a decisive role when it comes to creating shopping centers as the *future* consumption *place to be*. In a VUCA (volatility, uncertainty, complexity, ambiguity) environment, the targeted identification of added values is therefore essential. *The Value Creation Model in CREM/FM* is supportive with the identification of actual added values by integrating various stakeholder perspectives within the identification process. (Schlicht, 2018)

Figure 8: Value Creation Model in CREM/FM

Insights to *all* stakeholders' demands as well as to the *subordinate* corporate strategy are essential in terms of the identification of individual needs influencing the customer journey sustainably. This requires thinking outside-the-box in order to create great impact with the installation of *(at first sight)* small FM measures. Keeping in mind the effective communication of respective measures, customers will soon forget about online retail due to shopping experience options available offline and on site. This empirical study helped to generate additional important insights regarding the method of measuring the added value of FM as a competitive advantage and a knowledge generating tool for corporates.

5 Conclusion

Summing up the results from the research case it is important to have a clear understanding of goals, outperformance and what added value in FM is about. With regard to this the preliminary literature analysis showed clearly, that there is no common understanding of added value in FM, although a widely accepted definition is the essential basis for any comparable and reliable measurement. This goes along with the necessity of visible company goals to departments and adding value stakeholders. The real estate strategy has to be clearly specified and operationalized to all departments in a top-down manner. This can be ensured easily be connecting CREM and FM figures with KPIs from core business activities. But simply the connection of KPIs is not sufficient enough: communication needs to be operated additionally, which means that the top management needs to be involved within the development of the real estate strategy and the identification of influencing performance factors between function departments. For a broader acceptance of using the Track Record model to identify the added value of all participants, it is also useful to make success visible to employees, managers and further stakeholders by using visual presentation forms like the dashboard and data driven calculations. The presented model is therefore both: an assessment tool for the measurement of added value, but also a communication instrument for all participants in adding value to core business.

ACKNOWLEDGMENTS and Limitations

The idea of measuring added value in our presented form and the discussion of presented results are restricted by methodological content limitations. The conceptual development of the measurement model is based on the previously carried out quantitative study which contained 30 filled out questionnaires. A wider range of returns or additionally carried out studies would increase the statistical validity and specify the preliminary causal model in a more detailed way. Additionally, the heterogeneous definitions and terminologies of added value in FM could lead to a varying understanding and pervert results from a comprehensive point of view. Therefore, the model should be seen as a first prototype without pretense to be a complete structural equation model.

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IX

Measuring Innovation

Mario Hegedus

1 Introduction

The importance of measuring innovation, as well as a tendency or effort to measure them more effectively continuously increases. For a long time, in a professional community the question of appropriateness and reliability of particular methods and approaches in measuring innovation has been widely discussed. This growing interest in innovation measurement has been determined by gradual awareness and general acceptance of the importance of innovation at the national economy level, global economy level and for enterprises themselves.

The ambition of the present chapter is to briefly explain to a reader what is the innovation, what are the current trends in measuring innovation activity at the macroeconomic level and describe the methodology of measurement which is with the highest frequency applied in social practice.

The objective of the chapter is to briefly explain the concept of innovation and methods applied in its measurement. It deals with the macroeconomic level of measuring innovation activity and due to a complex character of the topics it is not analyzing the measuring of innovation at the enterprise (microeconomic) level.

2 A brief excursion into the history and definition of the innovation concept

A well- known founder of the theory of innovation was J.A. Schumpeter, a famous economist of an Austrian origin that later emmigrated to the USA. In his book *The theory of Economic Development* (for the first time published in 1911) he defined innovation (in his original terminology *new combinations* – the term innovation Schumpeter applied much more later) as inventions that are introduced to the market and bring to the entrepreneur-innovator a return in the form of supernormal profit, that is achieved up to the moment when imitators begin to take off the innovation and this leads to a decrease in the profit (Schumpeter 1987, pp. 195-197). As we can see, Schumpeter regarded innovation just as a first successful introduction to the market and he pointed out the key role of an entrepreneur in this process that was the source of innovation (Samuelson-Nordhaus 2003, p. 186). Any next introduction to the market was – according to Schumpeter – just an imitation. Incremental improvements in already established innovation were not- according to Schumpeter – new combinations, but represented just "a change in data".

In our paper we understand innovation in a more modern and less rigid form that includes into the innovation concept also incremental innovation and does not reduce innovation just to unique and the first introduction to the market (as it was the case in Schumpeter´s view).
This broader concept of innovation is applied also in *Oslo Manual* where we can find the following definition „An **innovation** is a new or improved product or process (or combination thereof) that differs significantly from the unit's previous products or processes and that has been made available to potential users (product) or brought into use by the unit (process)"(OECD – EUROSTAT, 2018, s. 20). Although the defining of innovation has been still discussed in literature, for the further analysis in our paper the definition of the innovation from Oslo Manual is used. In author´s opinion this material is a basic device for a macroeconomic measurement of innovation in the world and is accepted also by EUROSTAT, OECD, UNESCO, etc. The most important indicators and scoreboards for measuring innovation – such as OECD Science, Technology and Innovation (STI), European Innovation Scoreboard (EIS), Global Innovation Index (GII) – that are accepted by professional community, are also based on Oslo Manual´s methodology.

The further development of Oslo Manual has reflected also the new trends in the development of innovation models that are (together with the innovation theory) oriented to keep pace with the business practice and more generally to respond to the requirements of social needs and new challenges in the today´s world.

According to some authors [*N. du Preez a L. Louw* (2008), *C. Eleveens* (2010), *O' Raghallaigh et al.* (2011), *M. Kotesmir a D. Meissner* (2013)] there have been 5 generations of innovation models that were gradually developed as a response to particular needs and challenges in different periods. Recently the sixth model has been included - an open innovation model.

The open innovation concept is relatively new and its critical contribution and message is still to be fulfilled. It is logical that the model will respond to new trends - such as a sharing economy. According to M. Špaček the sharing economy development can significantly influence the future of an open innovation model and business models in general (Špaček 2017).

The sharing economy is expected to grow. According to PWC estimates the sales revenue of a sharing economy (as a part of global sales) will grow from $ 15 bill in 2015 to $335 bill in 2025 (PWC 2018).

The recent version of Oslo Manual (2018) has an ambition to respond to these new trends. Criteria quantifying the open innovation are included.

3 Development of the basic methodology to measure innovation – Oslo Manual

According to F, Gault (Gault, 2016) a precondition to measure innovation is to define the innovation for statistical purposes. A first codification of innovation definition for statistical purposes was in the first edition of Oslo Manual in 1992. It was based on statistical surveys performed in 1970s and 1980s. The first edition was concentrated mainly on the production sector, services were mentioned just in the section 239 and they were actually reduced to technological innovation of products and processes (a process being understood as a production process of products).

Five years later was Oslo Manual revised to include also services that achieved - at that time – the dominant share in GDP. Nevertheless just services connected with the technological innovation of products, processes and methods of introducing products to the market were included. Actually, innovation connected with introducing products to the market was added. This enlargement in the services interpretation was due to many factors that happened in the period between the first and second edition of Oslo Manual. The national accounts system has been revised, expenditures on software started to be regarded as capital investment, etc. An intensive discussion on the systemic approach to understand innovation were going on in OECD forum.

Simultaneously with discussing the measurement of innovation, the interest in organizational changes and business practices connected with the knowledge management has been growing as well. This led to an OECD project creating a platform for surveys of using knowledge management tools, on the basis of which a model questionnaire was created. An important new knowledge coming out from this research was that business practices could be regarded as a technology using similar measurement techniques to those used in the research of production technologies. These new results were incorporated into the next revised version of Oslo Manual (2016, p.1-2)

The innovativeness of the third version of Oslo Manual from the year 2018 characterizes OECD in a following way: "This new edition contains a number of major novelties, compared to the previous 2005 edition, to enhance the relevance of the manual as a source of conceptual and practical guidance for the provision of data, indicators and quantitative analyses on innovation. This manual:

· Provides a conceptual framework and a general definition of innovation that is applicable to all sectors in the economy (Business, Government, Non-profit institutions serving households and Households). These are necessary for developing future guidelines for measuring innovation in sectors other than business and eventually building up an economy- and society-wide statistical view of innovation, as recommended in the 2016 OECD Blue Sky Forum.

· Updates and streamlines core definitions and taxonomies to facilitate reporting and interpretation across the entire business sector, including service sector firms specialised in providing knowledge-based services.

· Supports the measurement of investment in intangible assets by making a link between intangibles (also described as knowledge-based capital) and the generation of different types of knowledge for innovation, providing explicit measurement recommendations.

· Provides guidance on measuring internal and external factors influencing business innovation, integrating previous ad hoc guidance on measuring innovation in developing countries, as well as addressing the need to measure the incidence and effect of diverse government policies on innovation.

· Promotes the collection of a broader set of data relevant to both non-innovative and innovation-active firms to help analyze the drivers and enablers of innovation.

· While the baseline definition of innovation in this manual does not require it to be a success, recommendations are provided for measuring attributes of the outcomes of innovation. This aims to facilitate a better understanding of the diverse range of innovations and and their impacts on the firm and the market and the broader social context in which it operates.

· Provides extended methodological guidelines for the entire innovation data lifecycle, from survey design and testing to data dissemination and curation. Compared to previous editions of the manual, there is considerably more guidance on methods for assessing question items and the implications of using different survey methods. The importance of the length of the observation period is discussed, highlighting the importance of seeking greater international convergence in survey practices.

· Extends guidance on the linkage of surveys with other sources, such as administrative records, and proposes complementary methods for obtaining evidence on a firm's focal (i.e. most important) innovation. Integrating an object- based approach can deliver significant improvements in survey data quality.

· Supports users of innovation data with a new chapter explaining the use of statistical data on innovation to construct indicators and for analysis. It presents a blueprint for the production of statistical indicators of innovation by thematic areas, drawing on the recommendations in previous chapters. It also describes methods for analysing innovation data, with a major focus on the analysis of innovation impacts and the empirical evaluation of innovation policies.

· Provides a glossary of key terms for ease of reference and to facilitate translation efforts to different languages.

In addition, this manual contributes to a better understanding of digitalization and its links with innovation by providing guidance on the role of digitized information from both a product and business process innovation perspective. It also achieves this goal by recognizing data development activities, along with software, as a potential innovation activity; highlighting data management competences as a key potential innovation capability for measurement, as well as recommending the measurement of external factors such as the role of digital platforms in the markets in which the firm operates.

The analysis of globalization and how it shapes innovation is supported by guidance on capturing knowledge flows with the rest of the world and the role of multinational enterprises (MNEs) and mapping the position of a firm's business processes within value chains. International coordination is called for when interpreting data on the role of MNEs" (2018, p. 21-23).

4 Criteria used for measuring innovation in OsloManual 2018

As several times mentioned in the previous part, Oslo Manual 2018 tries to better quantify innovation activity connected with acquiring, using and providing knowledge between firms and between firms and other organizations, that means the concept of an open innovation.

Basic indicators are summarized in Tables 1 to 6.

Table 1 Indicators of innovation incidence and characteristics

General topic	Indicator
Product innovations	Share of firms with one or more types of product innovations
New-to-market (NTM) product innovations	Share of firms with one or more NTM product innovations (can also focus on new-to-world product innovations)
Method of developing product innovations	Share of firms with one or more types of product innovations that developed these innovations through imitation, adaptation, collaboration, or entirely in-house
Other product innovation features	Depending on question items, indicators can capture attributes of product innovations (changes to function, design, experiences etc.)
Business process Innovations	Share of firms with one or more types of business process innovations
NTM business process innovations	Share of firms with one or more NTM business process innovations
Method of developing business process innovations	Share of firms with one or more types of business process innovations that developed these innovations through imitation, adaptation, collaboration, or entirely in-house
Product *and* business process innovations	Share of firms with both product and business process innovations
Innovative firms	Share of firms with at least one innovation of any type
Ongoing/abandoned innovation activities	Share of firms with ongoing innovation activities or with activities abandoned or put on hold
Innovation-active firms	Share of firms with one or more types of innovation activities

Table 2: Indicators of knowledge-based capital/innovation activities

General topic	Indicator
Knowledge-based capital (KBC) activities	Share of firms reporting KBC activities that are *potentially related to* innovation
KBC activities for innovation	Share of firms reporting KBC activities *for innovation*
Expenditures on KBC	Total expenditures on KBC activities *potentially related* to innovation
Expenditures on KBC for innovation	Total expenditures on KBC activities *for innovation*
Innovation expenditure share for each type of activity	Share of expenditures for innovation for each of seven types of innovation activities
Innovation expenditures by accounting category	Total expenditures for innovation activities by accounting category
Innovation projects	Number of innovation projects
Follow-on innovation activities	Share of firms with ongoing follow-on innovation activities
Innovation plans	Share of firms planning to increase (reduce) their innovation expenditures in the (current) next period

Source: OECD – EUROSTAT (2018, p. 224)

Table 3 Indicators of potential or actual innovation capabilities

General topic	Indicator
Innovation management	Share of firms adopting advanced general and innovation management practices
IP rights strategy	Share of firms using different types of IP rights
Workforce skills	Share of firms employing highly qualified personnel, by level of educational attainment or by fields of education
Advanced technology use	Share of firms using advanced, enabling or emerging technologies
Technical development	Share of firms developing advanced, enabling or emerging technologies

General topic	Indicator
Design capabilities	Share of firms with employees with design skills
Design centrality	Share of firms with design activity at different levels of strategic importance (Design Ladder)
Design thinking	Share of firms using design thinking tools and practices
Digital capabilities	Share of firms using advanced digital tools and methods
Digital platforms	Share of firms using digital platforms to sell or buy goods or services Share of firms providing digital platform services

Source: OECD – EUROSTAT (2018, p. 224-225)

Table 4 Indicators of knowledge flows and innovation

General topic	Indicator
Collaboration	Share of firms that collaborated with other parties on innovation activities (by type of partner or partner location)
Main collaboration partner	Share of firms indicating a given partner type as most important
Knowledge sources	Share of firms making use of a range of information sources
Licensing-out	Share of firms with outbound licensing activities
Knowledge services providers	Share of firms with a contract to develop products or business processes for other firms or organisations
Knowledge disclosure	Share of firms that disclosed useful knowledge for the product or business process innovations of other firms or organisations
Knowledge exchange with higher education institutions (HEIs) and public research institutions (PRIs)	Share of firms engaged in specific knowledge exchange activities with HEIs or PRIs
Challenges to knowledge exchange	Share of firms reporting barriers to interacting with other parties in the production or exchange of knowledge

Table 5 Indicators of external factors influencing innovation

General topic	Indicator
Customer type	Share of firms selling to specific types of customers (other businesses, government, consumers)
Geographic market	Share of firms selling products in international markets
Nature of competition	Share of firms reporting specific competition conditions that influence innovation
Standards	Share of firms engaged in standard setting activities
Social context for innovation	Share of firms reporting more than *N* social characteristics that are potentially conducive to innovation
Public support for innovation	Share of firms that received public support for the development or exploitation of innovations (by type of support)
Innovation drivers	Share of firms reporting selected items as a driver of innovation
Public infrastructure	Share of firms reporting selected types of infrastructure of high relevance to their innovation activities
Innovation barriers	Share of firms reporting selected items as barriers to innovation

Source: OECD – EUROSTAT (2018, p. 226)

Table 6 Indicators of innovation objectives and outcomes

General topic	Indicator
General business objectives	Share of firms reporting selected items as general objectives[1]
Innovation objectives	Share of firms reporting selected items as objectives for innovation activities[1]
Innovation outcomes	Shares of firms attaining a given objective through their innovation activity[1]
Sales from new products	Share of turnover from product innovations and new-to-market product innovations
Number of product innovations	Number of new products (median and average)
Changes to unit cost of sales	Share of firms reporting different levels of changes to unit costs from business process innovations
Innovation success	Share of firms reporting that innovations met expectations

Source: OECD – EUROSTAT (2018, p. 226)

Despite a short time after publishing the mentioned revised edition, the partial application of the new criteria can be seen e.g. in *European Innovation Scoreboard 2019,* where the new indicators are given in Table 7. EIS 2019 actually applies in practice the criteria suggested in Oslo Manual 2018. Of course, we can expect that due to a relatively long time needed to collect, process and evaluate several data, the new, revised OM criteria will be subsequently integrated into all basic scoreboards of innovation activity in member countries.

Table 7: EIS 2019 innovation indicators

FRAMEWORK CONDITIONS

- Human resources
  - 1.1.1 New doctorate graduates
  - 1.1.2 Population aged 25-34 with tertiary education
  - 1.1.3 Lifelong learning
- Attractive research systems
  - 1.2.1 International scientific co-publications
  - 1.2.2 Top 10% most cited publications
  - 1.2.3 Foreign doctorate students
- Innovation-friendly environment
  - 1.3.1 Broadband penetration
  - 1.3.2 Opportunity-driven entrepreneurship

INVESTMENTS

- Finance and support
  - 2.1.1 R&D expenditure in the public sector
  - 2.1.2 Venture capital expenditures
- Firm investments
  - 2.2.1 R&D expenditure in the business sector
  - 2.2.2 Non-R&D innovation expenditures
  - 2.2.3 Enterprises providing training to develop or upgrade ICT skills of their personnel

INNOVATION ACTIVITIES

- Innovators
  - 3.1.1 SMEs with product or process innovations
  - 3.1.2 SMEs with marketing or organisational innovations
  - 3.1.3 SMEs innovating in-house
- Linkages
  - 3.2.1 Innovative SMEs collaborating with others
  - 3.2.2 Public-private co-publications
  - 3.2.3 Private co-funding of public R&D expenditures
- Intellectual assets
  - 3.3.1 PCT patent applications
  - 3.3.2 Trademark applications
  - 3.3.3 Design applications

IMPACTS

- Employment impacts
  - 4.1.1 Employment in knowledge-intensive activities
  - 4.1.2 Employment fast-growing enterprises of innovative sectors
- Sales impacts
  - 4.2.1 Medium and high-tech product exports
  - 4.2.2 Knowledge-intensive services exports
  - 4.2.3 Sales of new-to-market and new-to-firm product innovations

Source: Hollanders, H. (2019, p. 4)

5 Conclusion

Measuring innovation activity is a demanding discipline which has been criticized by many authorities for insufficient ability to measure innovation activity by relevant variables and instead the measuring of innovation is to a high degree based on proxy variables that do not express the actual volume of innovation activity (Hall and Ziedonis 2001; Heller et al. 1998).

Another point of criticism regarding the validity of the mentioned innovation measuring models, is their unified approach to all countries, although in reality different countries find themselves in different stages of development. According to these critics the individual approach to evaluate innovation activity in different countries is necessary, taking into account the stage of their economic development (Beyhan et al.2009).

Irrespective of the discussion in literature regarding the correctness or incorrectness of the selected and generally applied indicators and methods of measuring innovation activity, we must be aware of the importance and necessity to reliably measure innovation activity and factors determining it. Theory of innovation is a permanently changing and developing discipline which makes a relevant and high quality of innovation measurement a very difficult task.

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Modeling the Knowledge Brokering within Culture-Based Urban Regeneration Processes

Mária Tajtáková
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1 Introduction

Knowledge is considered the dominant feature of nowadays social transformations associated with globalization as the world-wide integration of economic activity resulting into the development of knowledge economy (Besley, 2010). Kelemen et al. (2010, p. 141) highlight that knowledge workers represent already more than half of the employees in advanced economies. In addition, Horibe (1999, p. x, xi) claims that in the New Economy – the economy based on the flood of information coming at lightning speed – the demand at the workplace is almost exclusively for knowledge workers.

All knowledge workers possess certain knowledge which they may share and exchange with others. However, as pointed out by Asrar-ul-Haq and Anwar (2016, p. 2) knowledge that is not well managed and shared corrodes easily. Especially, the tacit knowledge that resides in the minds of people and is accumulated over time must be shared. Hence, the knowledge sharing is considered to be the crucial element for an effective knowledge management. Here, the most important role is attributed to knowledge brokers (Tajtáková and Olejárová, 2019).

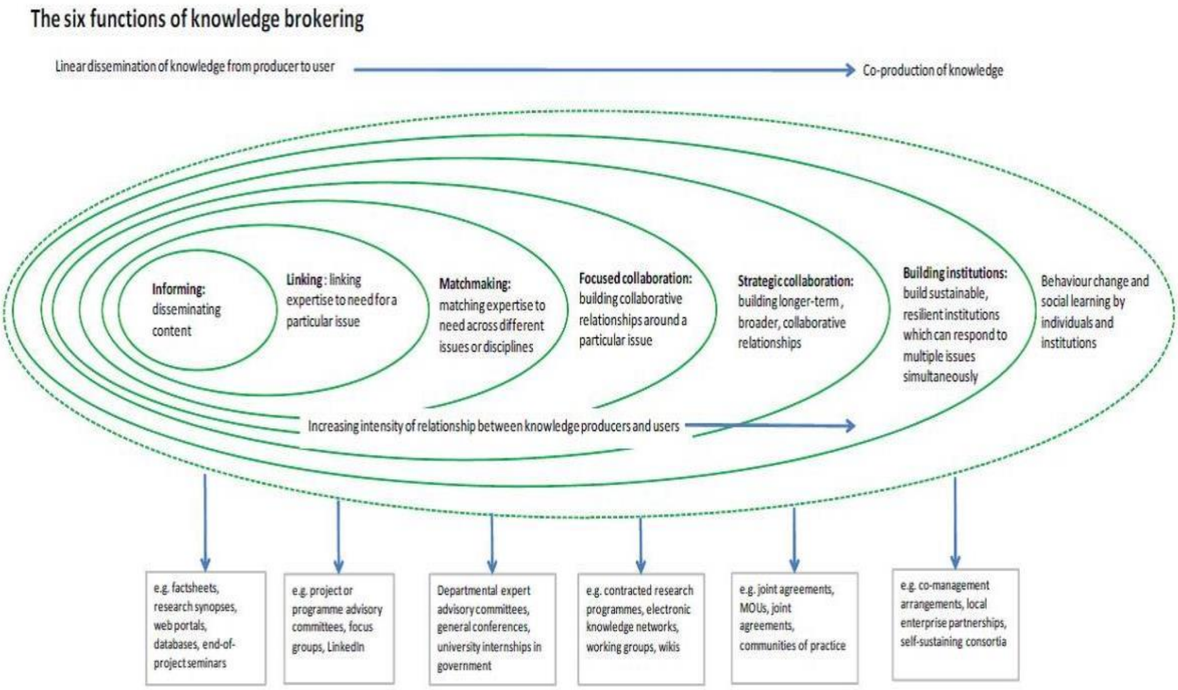
The knowledge brokers are defined as individuals or organizations that facilitate the sharing of different kinds of knowledge between knowledge sources and knowledge needs (Soussa, 2008). A distinction between knowledge brokers and most other knowledge service providers (e.g. consulting companies, state agencies, business intelligence firms) lies in their active role in the transformation processes. The knowledge brokering is used to improve the process of knowledge transfer and sharing among different members and actors in the network.

As pointed out by Porumb and Ivanova (2014) the knowledge distribution among diverse actors can be the most sensitive aspect of the knowledge management practice. Knowledge brokers act as catalysts, accelerating the combination of complementary knowledge and skills necessary to solve innovation problems, by making the right connections and links with solvers and seekers (Soussa, 2008). In this regard, Karner et al. (2014) highlight the knowledge brokerage as a way to link different perspectives, levels of knowledge and understandings among people. In doing this, the knowledge brokers may adopt several strategies.

2 Strategies of Knowledge Brokers within the Knowledge Transfer

Sarah Michael (2009, p. 994-1011) identified six different strategies of knowledge brokers as *informing, consulting, matchmaking, engaging, collaborating* and *building adaptive capacity*. Each strategy has a complementary function to the others and reflects a different stage in the knowledge brokering process. They are listed in order of increasing intensity of relationship building and commitment of resources so that the strategies that involve more effort subsume those that involve less.

Karner (2010, p. 14-15) points out that while informing, consulting, matchmaking require quite a low level of involvement, engaging, collaborating and capacity building need higher level of engagement and personal interaction in order to be effective. Shaxson and Gwyn et al. (2010, p. 4) implemented the six strategies of knowledge brokers and developed a schema (Figure 1) for knowledge translation and brokering in public policy making. The adjusted



functions of knowledge brokers include *informing, linking, matchmaking, focused collaboration, strategic collaboration* and *building sustainable institutions*.

Fig. 1 The Six Functions of Knowledge Brokers
Source: Shaxson, L. and Gwyn, E. et al. 2010, p. 4. Adaptation from Michaels, S. 2009

- **Informing**

Disseminating content, targeting decision makers with information, making information easily accessible and digestible. Examples include factsheets, research synopses, web portals, databases, end-of-project seminars.

- **Linking**

Linking expertise to need for a particular policy area, helping policymakers address a specific policy issue by seeking out the necessary experts. Examples include project or program advisory committees, focus groups, LinkedIn.

- **Matchmaking**

Matching expertise to need across issues and disciplines, helping policymakers think more broadly about a topic, finding experts with relevant knowledge from another discipline, helping them take a strategic overview to address the fullness of the issue. Examples include Departmental expert advisory committees, general conferences, university internships in government, mapping the evidence base for an issue.

- **Focused collaboration**

Beginning to construct formal relationships to focus on a particular issue, contracting people or organizations to provide knowledge on an as-needed basis. Examples include contracted research programs, electronic knowledge networks, working groups, wikis.

- **Strategic collaboration**

Lengthening and deepening the collaborative process, strengthening relationships and moving to a situation where all sides jointly negotiate the questions to be asked. Examples include joint agreements where the emphasis is on equality in the relationships between actors such as joint agreements and communities of practice.

- **Building sustainable institutions**

Deepening the collaborative relationship to the extent that all parties jointly frame the issue; broadening institutional capacity of institutions to respond to several issues simultaneously. The focus is on co-production of knowledge and joint learning from doing; the arrangements are self-sustaining in terms of both funding and function, with all sides contributing resources. Examples include co-management arrangements, local enterprise partnerships, self-sustaining consortia.

3 Culture-based Urban Regeneration and Knowledge Management

With the emergence of knowledge economy, the implementation of knowledge management has become increasingly important in diverse domains of the society. Our chapter focuses on the knowledge management application in the field of culture-based urban regeneration. The culture-based urban regeneration refers to the urban development effort using culture as a mean for economic grow, local community enhancement, social inclusion, and cultural tourism development leading to an overall urban and social revitalization.

Numerous reported cases of culture-based urban regeneration projects (e.g. Evans, G. 2005; Miles and Paddison, 2005; Nivin and Plettner, 2009; Tajtáková, 2010) demonstrate positive synergic effects between culture and economics within the urban development. Successful stories include cities like Glasgow, Liverpool, Manchester, Bilbao, Rotterdam, Dusseldorf, Cleveland, Pittsburgh and others. However, as pointed out by Pastak and Kährik (2016, p. 967), while in Western countries urban renewal has been on the political agenda for many decades, in Central and Eastern Europe, revitalization projects have been emerging only since the 2000s. Nevertheless, the experience from Western Europe suggests that well-designed cultural projects may turn the image of grey post-industrial places with nothing to offer into new cultural venues attracting tourists and private investors and encouraging young people and entrepreneurs to stay or to return to create their own businesses.

Borg and Russo (2005, p. 28-29) identified three main "impact areas" of culture on urban development, which they further integrated into the *Culture-Oriented Economic Development* (COED) model for the city (Figure 2). The model includes:

- Direct economic impact of employment and value generation in cultural industries and indirect expenditure effects related to cultural professions.
- Induced effects of cultural activities on the quality of a place encompassing the tourist attractiveness, which leverages additional visitor expenditure, but also the location amenities for companies.

- “Creative inputs” accruing to the local networks of production (both to products and to processes of production or organizational models). These are “cultivated” in a lively and stimulating cultural environment, where a creative class develops, attracted by tolerance, openness, educational and social opportunities.

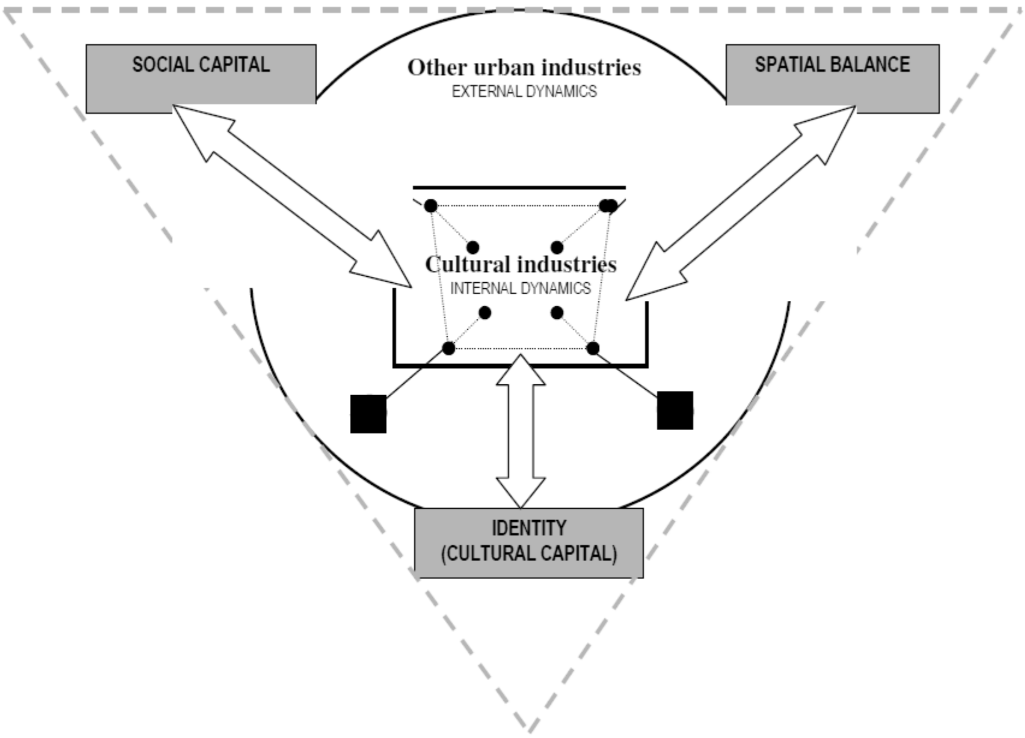


Fig. 2 Culture-Oriented Economic Development (COED) model for the city
Source: Borg and Russo, 2005, p. 32

The authors suggest to utilize the COED model as a reference framework to evaluate the role and effects of culture on the economic development of cities considering the “economic strength” of the cultural cluster, the “fertilization” of the local economic milieu by culture and creativity, and the “sustainability” of the process of economic growth determined by inclusive cultural activities and projects, their accessibility and spatial distribution in the urban area (Borg and Russo, 2005, p. 32-33).

We assume that the crucial factor determining the success of COED model within the context of culture-based urban development and the factor connecting all three areas is the effective knowledge management. We believe that the expected economic effects, community inclusion and cultural sustainability within urban development can only be achieved in the repeated cycles of knowledge creation, externalization, sharing, and critical assessment. In this respect, Bureš

(2009) highlights a unique value and character of knowledge as it possesses such attributes which are absent in other resources. The bearers and disseminators of knowledge are above all the knowledge brokers. In the further text we will focus on modeling the knowledge brokering within culture-based urban regeneration processes.

4 Methodology

The aim of the study was to identify, analyze, compare and summarize best practices in using knowledge management tools for an effective and efficient knowledge transfer in culture-based urban regeneration projects in Slovakia.

The study was based on the qualitative research method, with the use of in-depth personal interviews carried out with the managers and leading personalities of selected organizations involved in urban revitalization projects in Slovakia. In total, seven managers/creative leaders were interviewed. In order to search for similarities and/or differences within the identification of successful knowledge practices, it was crucial to use a set of identical questions through all studied categories. Our intention was to analyze, compare and thus to identify successful practice based on knowledge transfer and sharing.

Moreover, the purpose of the used method was to obtain opinions of creative and innovative actors, recognize their know-how and collect sufficient amount of data for creating a learning model. Our interest was also concentrated on indicating internal and external factors, opportunities and barriers. The existence of creative and innovative elements distinguishing the leaders (knowledge brokers) and their teams from the others and contributing to obvious success was a part of our research as well. Primary qualitative research was complemented by the secondary research based on available printed and electronic resources from the revitalization projects and official public reports.

The sample consisted of six organizations located in three different Slovak cities (Table 1): Kultúrne centrum – KC Dunaj (Cultural Centre Dunaj) – Bratislava, Stará tržnica (Market City Hall) – Bratislava, Stanica Žilina-Záriečie (Local Train Station) – Žilina, Nová Synagóga (New Synagogue) – Žilina, Intercity (IC) Culture Train – Vyšné Opátske, and Tabačka (Tobacco Factory) Kulturfabrik – Košice.

Findings were processed using the methods of description, analysis, induction and comparison of existing approaches towards knowledge management in examined culture-based urban regeneration projects. Moreover, the SWOT analysis of innovative and creative approaches focusing on culture-based urban regeneration processes enabled us to highlight effective tools and methods in order to formulate criteria and conclusions. Afterwards, the perspective of two different knowledge management models – adjusted SECI model and Knowledge Broker Intervention Model – was applied.

Tab. 1 The Sample

	Revitalization project	**City**	**Original purpose vs. new purpose of the venue**	**Revitalization period**
1	KC Dunaj	Bratislava	Department Store – Independent Cultural Centre	2010, ongoing
2	Stará tržnica	Bratislava	Market City Hall – Cultural Center & Market Hall	2013, ongoing
3	Stanica Záriečie	Žilina	Local Train Station – Independent Cultural Center	2003, ongoing
4	Synagóga	Žilina	New Synagogue – Cultural & Community Centre	2011, ongoing
5	IC Culture Train	Košice	Suburb Community Centre – Independent Cultural Centre	2005 – 2008
6	Tabačka Kulturfabrik	Košice	Tobacco Factory – Independent Cultural Centre & Creative Incubator	2009, ongoing

Source: Own processing

5 Modeling the Knowledge Brokering within Urban Regeneration Processes

5.1 The SECI Knowledge Management Model

The starting point for modeling the knowledge brokering within culture-based urban regeneration processes was the adapted SECI knowledge management model (Figure 3). The model is grounded in Nonaka and Takeuchi's SECI model encompassing four quadrants/stages of knowledge transfer, namely socialization, externalization, combination and internalization. It suggests that knowledge sharing can support the creation of new knowledge since it can transfer both tacit knowledge and explicit knowledge into organizational knowledge (Nonaka and Takeuchi, 1996). The SECI model comprises most decisive factors necessary for an effective knowledge management cycle encompassing the creation of knowledge, its expression, sharing, transfer and critical assessment.

The SECI model highlights the need of assessing all consecutive stages of knowledge conversion in order to achieve a desired goal: the efficient and effective knowledge management and communication among all actors (team leaders, team members and stakeholders). Hvorecký (2011) emphasizes that the SECI model helps to share, transfer, combine and implement valuable experiences and skills and thus reuse and refine them in order to create a competitive advantage.

Jin-Feng et al. (2017) argue that it is above all tacit knowledge which has become one of the important sources of organizational core competitiveness, on account of its imperfectly replicable and imitable attribute. The authors highlight that the tacit knowledge comes from the gradual enlightenment and insight during the working process and is related to the experiences stored in the minds of employees. This kind of tacit knowledge is difficult to obtain, transfer, share and manage. The tacit knowledge appears especially important in the field of culture-based urban regeneration. Effective knowledge brokering is therefore one of the most important success factors in this particular area.

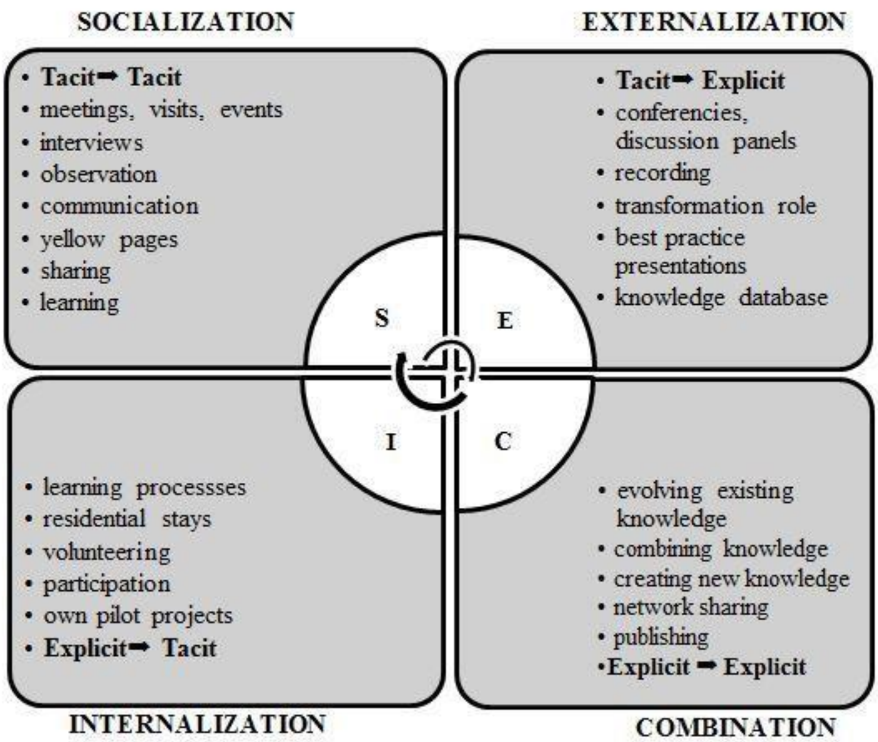


Fig. 3 Adjusted SECI model of knowledge management for culture-based urban regeneration

Source: own processing based on the Nonaka and Takeuchi's (1996) SECI model schema

Model elements

- **Socialization**

The initial stage of the SECI model runs within the context of culture-based regeneration projects on several levels:

1. Among the team members involved in regeneration projects (internally) and all stakeholders (externally) on local and regional levels.
2. Among the teams of non-profit organizations registered in the Slovak network of independent cultural centers (ANTÉNA) on the national level.
3. Among the members of the international network (T.E.H. – Tran Europe Halles), as the oldest and largest organization assisting in sharing experiences among European independent cultural centers on the international level.

To illustrate the most effective and frequently used tools we can mention visits, meetings, stays, and presentations focused on identifying, learning and sharing best practice of revitalization projects.

- **Externalization**

The stage of externalization includes the organization of meetings, workshops, conferences, and panel discussions with the use of explicit visual forms of knowledge management as pictures, videos, photographs, data presentations, statistics and reviews of visitors, or even documentaries from the reconstruction phases of the revitalized venues. Since most of the urban regeneration projects are run by non-profit organizations supported via public funding schemes the externalized knowledge is mandatory available also on their websites as a tool for ensuring the transparent operations within the subsidized projects.

- **Combination**

Authors assume that the potential of the combination stage lies in the implementation of information and knowledge exchange platforms, inspiration, adoption of successful ideas, and the creation of new ones. However, it is important to be aware that universal rules do not always work and specific local or regional differences should be considered. Therefore it is recommended to achieve a high diversity and flexibility in activities such as program offers, entire dramaturgy, seasonal character, stakeholders' input and volunteers' involvement.

- **Internalization**

The final stage of the SECI model – internalization – appears to be an opportunity for creating a certain added value to all previous stages. Within this phase it is expected to implement the skills and knowledges developed during the effective learning processes in previous stages, active participation or creative and innovative approach. Through such activities as workshops, residential stays, visits, volunteering, and regular communication, the project teams shall succeed in developing and managing their own revitalization projects, with all signs of transparency, effectiveness and sustainability.

5.2 Knowledge Broker Intervention Model

Knowledge brokering is a strategy, approach or process that facilitates the exchange of knowledge between producers and users. It serves two purposes: first, to improve the utility of knowledge so that it actively informs decision-making and has a noticeable effect on the quality of decisions, policies and processes. Second, it aims to improve the receptivity of decision-makers to new knowledge (Shaxson and Gwyn et al., 2010, p. 6).

Thus, the knowledge brokering is used to improve the process of knowledge transfer and sharing among different members and actors in the network. However, as emphasized by Porumb and Ivanova (2014) the most sensitive aspect of the knowledge management practice is the knowledge distribution among diverse actors. Therefore, to facilitate the knowledge distribution the authors suggest implementing the Knowledge Broker Intervention Model (KBIM) (Figure 4).

The model is used to improve the networks of components and ensure suitable development by several consecutive steps of knowledge distribution, enhancement, appropriation and sharing. It regards people and makes possible the creation of productive and dynamic relationships aimed at facilitating the movement of ideas (Porumb and Ivanova, 2014).

The main role in the model is attributed to knowledge brokers defined as entities (organizations and individuals) that facilitate the sharing of different kinds of knowledge between knowledge sources and knowledge needs (Sousa, 2008). The knowledge brokers may have multiple types of power, such as legal authority, expert, referent, network, relational and catalytically; and must ensure four goals of knowledge exchange. They main tasks and competencies include (Porumb and Ivanova, 2014):

- Facilitate effective research cooperation;
- Guide the strategic direction of an organization;
- Enhance communication networks;
- Support effective dissemination and uptake of new knowledge.

In terms of process, the model builds on six different strategies of knowledge brokers identified by Sarah Michael (2009, p. 994-1011) as informing, consulting, matchmaking, engaging, collaborating and building adaptive capacity. Each strategy has a complementary function to the others and reflects a different stage in knowledge brokering process.

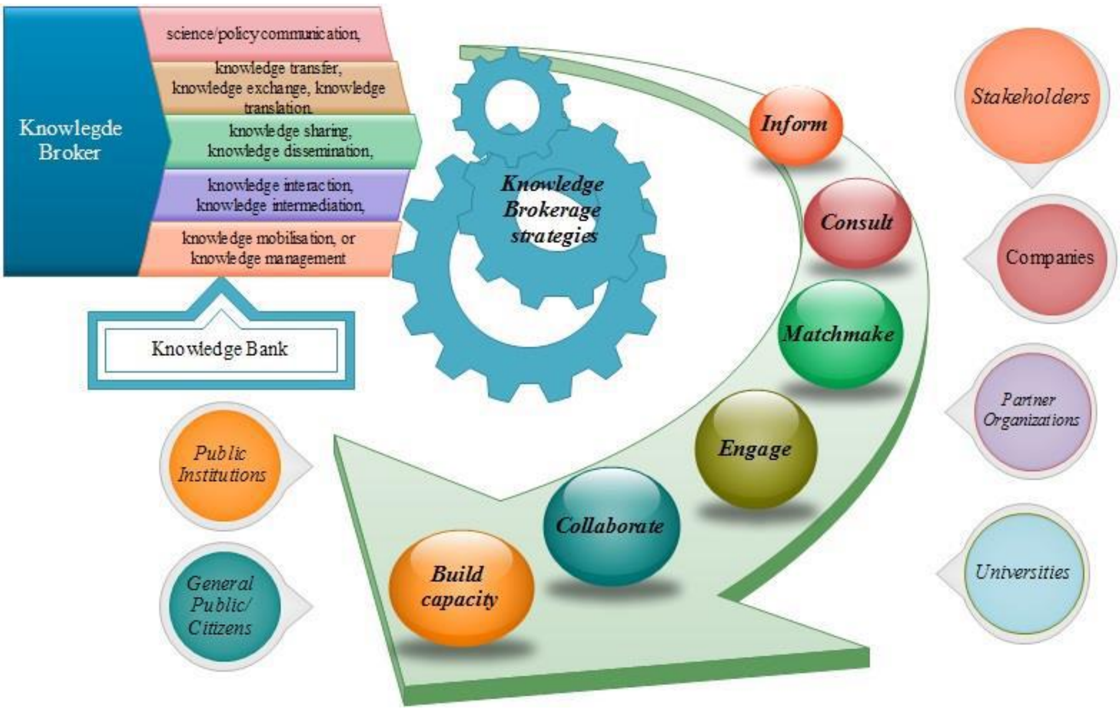


Fig. 4 Knowledge Broker Intervention Management (KBINM)
Source: Porumb and Ivanova, 2014, p. 456

Model elements

- **Informing**

Informing stands for disseminating the content, making it accessible and comprehensible to the recipients. It is expected that the knowledge broker packages the necessary information in a form, such as fact sheets, research synopses, databases, end-of-projects seminars, best practices and others, so that the recipient is able to understand it, evaluate what actions are needed, and then decide whether or not to take those actions.

- **Consulting**

Consulting involves linking expertise to needs in a specific area (e.g. particular policy area, helping policy makers address a specific policy issue by seeking out the necessary experts). The knowledge broker can be an intermediary, who is able to identify the type of expertise necessary to help solve the problem, and who establishes a connection.

- **Matchmaking**

Matchmaking brings together individuals able to deal with envisaged issues. It refers to matching know-how to needs across issues and subject matters, helping policy makers (or other knowledge users) think more generally about a topic, finding experts with relevant knowledge in required fields, helping them to come up with alternative perspectives in addressing the details of the issue. Through brokerage the actors are brought together: Building on the previous step (consulting) the knowledge brokers need to identify what expertise is needed, and then identify who can provide it in order to connect these people. Common platforms for matchmaking include advisory committees, conferences, seminars, workshops.

- **Engaging**

Engaging involves the party who is responsible for addressing the problem establishing and implementing the process of including others with salient expertise. The knowledge broker may play a role of a connector and facilitator. In this stage policy makers interact with individuals who have contradictory and complementary expertise. Examples represent expert networks, working groups, contracted research programs, electronic knowledge networks.

- **Collaborating**

Collaboration implies that all participants mutually develop a process through which they can interact with one another, negotiate and address a common problem or question. It requires that those engaged think beyond providing their knowledge, and that they reflect on how it can productively and usefully be put together with the expertize of others. Yet, while co-operators bring their expertise they must work as team members to deal with the complex issue at hand. Examples include joint agreements ensuring the equality in the relationships between different actors and communities of practice.

- **Building capacity**

Capacity building is understood as increasing the ability of people and institutions to do what is expected of them to do. In knowledge broker sense, building capacity presumes that parties jointly frame the process of interaction and negotiate substance with intent of addressing multiple dimensions of an issue, related concerns, lessons learned and future scenarios. Examples encompass co-management arrangements, local enterprise partnerships, and self-sustaining associations.

6 Results

The projects included into the sample reflect creative and innovative approaches aimed at achieving sustainability of the accomplished culture-based urban regeneration projects. Our research confirmed the significant role of project leaders who were acting as knowledge brokers. In total, three knowledge brokers were identified. They were involved in more than one project team (usually two projects) and transferred their knowledge gained in one project to another project. Especially, the transfer of tacit knowledge from different sources was significant for achieving a successful urban regeneration.

The projects which succeeded in the regeneration effort were characterized by clear adaptability, unique collaboration and venue diversity. In addition, Olejárová (2017) highlighted open communication with stakeholders, continuous feedback from users, transparent financing methods and sustainability concerns as common procedures of the innovative and creative management.

These findings indicate the presence of social innovation characterized by Tajtáková and Ferenčíková (2017) as a novel solution to a problem which is more effective, efficient and sustainable than current solutions relying on the inventiveness from citizens, civil society organizations, local communities, businesses, and public institutions. As emphasized by Porumb and Ivanova (2014, p. 8) knowledge brokers can enable innovation because they have access to knowledge from different territories, while increasing the amount of knowledge by making connections with a wide range of actors.

In order to track the role of the innovative and creative knowledge brokers – representing at the same time change agents – we carried out several in-depth interviews with people involved in diverse projects. After monitoring, comparing, analyzing, and evaluating the activities and transformation processes of knowledge brokers we can confirm the usage of knowledge management tools and methods within several frameworks. Based on the gathered data we identified common features of the knowledge brokers in the examined projects.

All knowledge brokers possessed previous skills and experiences from the former regeneration projects. Their expertise encompassed mainly following fields: reconstruction works, finance, human resource management, building and program scope, and cooperation with stakeholders. As a result, several specific roles of knowledge brokers were defined. The knowledge brokers were able to:

- Identify and attract people who are interested and want to be involved;
- Create, maintain, and nurture the knowledge bank;
- Build and keep relationships with stakeholders;
- Participate in existing networks to learn new information, knowledge and ideas;
- Create platforms and networks to share gained knowledge and best practices;
- Consult and cooperate with experts.

The knowledge brokers were particularly successful in attracting and involving other individuals, activists, and experts who either became team members, regular or occasional participants, or advisors of the projects. A know how gained within the culture-based urban regeneration projects became a valuable and stimulating base of knowledge not only for them but also for other peer individuals and organizations. A visible enthusiasm, creative and innovative approach were significant for all knowledge brokers and their teams, and resulted in building and strengthening long-term and supportive relationships with different stakeholders (communities, city and regional structures, donors, enterprises, institutions and volunteers). The role of the knowledge brokers involved also expectations and demands for continuous improvements.

In order to share gained knowledge, the knowledge brokers regularly took part in workshops, conferences and seminars. Moreover, it appeared very useful to participate in diverse professional platforms and networks (e.g. ANTÉNA, T.E.H. – Tran Europe Halles) providing opportunities for presenting skills and exchanging experiences, and thus help and encourage other organizations. Such knowledge platforms do not only mean a significant opportunity to learn about other successful projects, but also enable establishing cooperation, bringing experts together, and providing spill-over effect.

The opportunities for exchanging knowledge connect experts, managers, community workers, students and others to get inspiration, and at the same time to inspire the others through activities as residential stays, pilot projects and volunteering. The intention was also to support and connect communities, local and neighboring creative industries, students, small producers and farmers.

All case studies represented unique regeneration projects, the intention of which was not only to save unused urban buildings, but to revitalize them and convert them into attractive and sustainable cultural centers. Their representatives – the creative and innovative knowledge brokers – managed not only to achieve goals of the revitalization but also to gain respect and support of their stakeholders. The knowledge brokers received several prestigious awards and invitations to the cooperation on future urban regeneration projects, what can be seen as an evidence of the correct approach and appropriate knowledge management. An effective knowledge transfer led to the development of functional and sustainable multi-source financing schemes of regeneration projects and opened a multi-level communication with different actors.

7 Conclusions

Numerous attempts aimed at modeling the knowledge management approaches agree on a multidisciplinary paradigm, based on mapping processes with the use of knowledge database, including human mind and application of technologies. Appropriate methods of creating, utilizing, preserving, and disseminating knowledge enable to maintain desired organizational performance, and contributes to increasing quality and improving competitive advantage.

In this chapter we focused on the field of culture-based urban regeneration in view of the knowledge brokerage models application. The basis for the analysis was the SECI model including phases of socialization, externalization, combination, and internalization. During the consecutive stages the knowledge brokers were able not only to gain, create and share knowledge, but also to collaborate with stakeholders and build sustainability. Further, we enlarged our perspective by applying the Knowledge Broker Intervention model based on six strategies of knowledge brokers defined by Michael (2009) and further adapted by Shaxson and Gwyn et al. (2010) in order to analyze specific roles, tasks and competencies of knowledge brokers in the examined area.

The purpose of implementing the adapted SECI model was to demonstrate how to learn and benefit from best practices within the knowledge transfer and how to coordinate participative management of culture-based urban regeneration projects based on common effort involving the non-for-profit and cultural sector, government, businesses, media, target groups and volunteers.

The use of the Knowledge Broker Intervention Model was supported by the nature of culture-led urban regeneration practices. These are usually based on innovative and creative ideas carried out as bottom-up initiatives and managed by strong leaders. The leaders can further act as knowledge brokers, share and distribute their knowledge towards new or parallel culture-based urban regeneration projects.

As pointed out, the main role of knowledge brokers resides in the effective knowledge transfer and management between knowledge banks and knowledge needs. Besides different strategies for effective knowledge transfer and sharing, the knowledge brokers are challenged by expectations and demands for continuous improvements. Therefore, knowledge brokers – as change agents – shall not only present their experiences and skills, but shall also participate as active members in local, national and international networks and institutions.

To conclude, the identified knowledge brokers demonstrated the ability to learn and share valuable knowledge and experiences not only from their previous projects but also from the interaction with other national or international peers. They also demonstrated an effective knowledge transfer and management between knowledge banks and places of knowledge needs. The six functions of knowledge brokers were performed by the leaders of culture-based urban regeneration projects who shared and distributed their knowledge towards new or parallel endeavors. Both the regeneration itself and the entire knowledge management of the revitalization projects, including effective communication and knowledge transfer, deserve to be called a success story.

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Annex

Tab. 2 Analysis of Studied Categories in Case Studies

Studied subject	**KC Dunaj**	**Stará tržnica**	**Žilina Záriečie**	**Nová Synagóga**	**IC CT, Vyšné Opátske**	**Tabačka Kultur-fabrik**
Location	Bratislava	Bratislava	Žilina	Žilina	Košice	Košice
Actors & activities	Creators, Activists, Knowledge workers Reconstruct. the of 4th floor	Creators, Activists, Knowledge workers Complex Building Reconstruct.	Creators, Activists, Knowledge workers Building Reconstr. & Extention	Creators, Activists, Knowledge workers Historical Research, Restaura-tion & Reconstruct.	Creators, Activists, Knowledge workers Complex Building Reconstruct.	Creators, Activists, Knowledge workers Complex Building Reconstruct.

	Independent Culture Events	Multi- genre Events	Multi- genre Events	Half- openings& Cult. Events	Independent Culture Events	Multi- genre Events
History& Venue revitaliza -tion	Original Department Store	Original City Market Hall	Local Train Station	Synagogue	Suburb Center	Tobacco Factory
Venue purpose	Independent Cultural Center	Urban Cultural Center& Market Hall	Independent Cultural Center	Kunsthalle, Cultural Center	Independent Cultural Center	Independent Cultural Center
Program offer	IndependentMulti-genre Culture, Workshops, Discussions, Films, Talks, Dancing	Multi- genre Culture, Markets , (Non)public Events, Education	Independent multi-genre Culture, Creative Workshops for Pupils	Concerts, Exhibitions, Auctions, Talks, Seasonal Markets	Independent Multi-genre Culture	Independent Multi-genre Culture, Creative LAB, Cinema, Hostel, Workshops
Target groups	25-40yrs, Non-profit Sector, Tourists	Youth, Families, Seniors, Businesses, Tourists	Local: Children, Families, 18-40yrs, Tourists	Local: 20-45yrs, Businesses, Artists	Local: 20–40yrs, Children, Artists	Local: 18–45yrs, Artists, Tourists
Executive team	G. Bindics, KC Dunaj Team	Civic association STARÁ TRŽNICA	Civic association Truc sphérique	Civic association Truc sphérique	Civic association BONA FIDE	Civic association BONA FIDE
Financing structure	Bar& Restaurant, Events Fees, Rental	External Services, Markets, Events, Rental	Grants, Event Fees, Rental, Bar& Restaurant, Volunteers	Funds, Grants, Collections, Volunteer Assistance	Region& City Support, Event Fees	Bar&Bistro Region& City Support Arts Fund Event Fees, Rental
Stake- holders'	Community Creation,	Reputation,	Communíty Creation,		Community Growth,	Reputation,

feedback	Artist Support, High & Regular Visitors Network Membership, Non-profit Organization Support	Communíty Creation, Increasing Visit Rate Sectors Support, Volunteers Activities, City Award	Reputation, Foreign Support, Volunteers Activities, Networking, Foreign Award	Public Credit, Foreign Support, Business Support, Volunteers Activities, Community Growth, Local Award	City Support, Independent Culture Develop-ment ECC2013 Reputation, Networks	Region& Ministry of Culture Support, Funds, Networks, Volunteers, Local Award, Cross- sector Cooperation

Source: Own processing

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